# MATHEMATICS MAGAZINE

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# MATHEMATICS MAGAZINE

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# MATHEMATICS MAGAZINE

ROY DUBISCH, *EDITOR*




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# DISJOINT TRIANGLES IN CHROMATIC GRAPHS

J. W. MOON, University of Alberta

Let there be given $n$ distinct points in a plane. If each pair of points is joined by an edge and if each edge is colored either red or blue, then the resulting configuration is called a *chromatic graph* $G_n$. A *monochromatic triangle* is a subset of three points of $G_n$ such that the three edges joining these points all have the same color. The problem of determining the minimum number of monochromatic triangles a chromatic graph can have has been considered in several recent papers (cf. [2], [6] and [4]). (Analogous problems for chromatic bipartite graphs were considered in [5] and [1].)

The object in this note is to derive simple bounds for $\mu(G_n)$, the maximum number of mutually disjoint monochromatic triangles in the chromatic graph $G_n$. (Two triangles $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ are disjoint if $x_i \neq y_j$ for $i, j=1, 2, 3$.) It follows from our results that, roughly speaking, $\mu(G_n)$ depends almost entirely on the value of $n$ and very little on the actual structure of the graph $G_n$.

If any point $a$ of the graph $G_n$ is incident with three edges of the same color, say $(a, x)$, $(a, y)$ and $(a, z)$, then $G_n$ certainly contains at least one monochromatic triangle. (Consider the possible ways of coloring the three edges joining the points $x$, $y$, and $z$.) The following well-known facts (see, e.g., [3]) are easy consequences of this observation.

A: $\mu(G_6) \geqq 1$ for any chromatic graph $G_6$.

B: If $\mu(G_5)=0$, then there are two edges of each color incident with each point of $G_5$.

We now prove the following result where, as usual, $[x]$ denotes the greatest integer not exceeding $x$.

THEOREM. *If $G_n$ is any chromatic graph with $n$ points, then*

$$[\tfrac{1}{3}n] - 1 \leqq \mu(G_n) \leqq [\tfrac{1}{3}n]; \tag{1}$$

*furthermore, if $n \equiv 2 \pmod 3$ and $n \geqq 8$, then*

$$\mu(G_n) = [\tfrac{1}{3}n]. \tag{2}$$

*Proof.* Suppose that $\mu(G_n)=k$ and let $T$ denote some maximal family of $k$ mutually disjoint monochromatic triangles of $G_n$. It is obvious that $k \leqq [\frac{1}{3}n]$ since each triangle involves three points and no point belongs to more than one triangle in $T$. If $k \leqq [\frac{1}{3}n]-2$, then there are

$$n - 3k \geqq n - 3[\tfrac{1}{3}n] + 6 \geqq 6$$

points not belonging to any triangle in $T$. According to (A) there exists at least one monochromatic triangle involving three of these leftover points. This triangle and those in $T$ form a set of $k+1$ mutually disjoint monochromatic triangles. This contradicts the assumption that $\mu(G_n)=k$ and suffices to complete the proof of (1).

We now prove (2) for the case $n=8$. It is clear that $1 \leqq \mu(G_8) \leqq 2$ for any graph $G_8$. Let us suppose that $\mu(G_8)=1$ for some graph $G_8$. Then we may apply

(B) to the five nodes not involved in any given monochromatic triangle of $G_8$. We may therefore assume, without loss of generality, that if $\mu(G_8)=1$ then $G_8$ contains the subgraph indicated in Figure 1. (Solid lines denote red edges and dotted lines denote blue edges.)

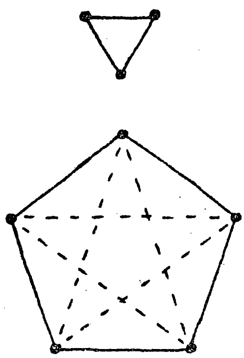

FIG. 1.

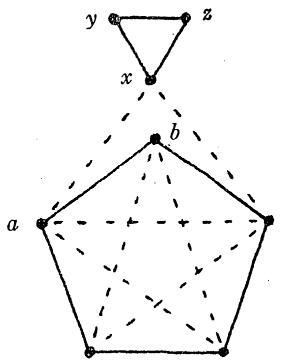


FIG. 2.

Let us suppose that one of the points of the triangle, say $x$, and two of the points of the pentagon, say $a$ and $c$, form a blue triangle. (See Figure 2.) If we apply (B) to the points other than $a$, $c$, and $x$ we see that the point $b$ must be joined to two additional points by red lines. The points $y$ and $z$ are the only ones available. But then $G_8$ would contain two disjoint monochromatic triangles, $(a, x, c)$ and $(b, y, z)$.

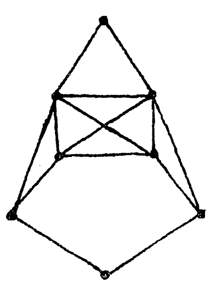

FIG. 3.

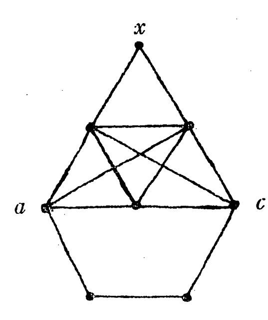


FIG. 4.

It is not difficult to see that each point of the triangle in Figure 1 must be joined by a red edge to (at least) three consecutive points of the pentagon if there are to exist no blue triangles of the type $(a, x, c)$. It follows, therefore, that at least two points of the triangle are such that the two sets of three consecutive nodes of the pentagon joined to these points by red edges have at least two points in common. Consequently, we may assume that $G_8$ contains one of the subgraphs indicated in Figures 3 and 4. (The dotted lines have been omitted.)

The first possibility can be excluded because $G_8$ would then contain two disjoint red triangles. The second possibility can be excluded also, because the point $x$ must be joined to one of the points $a$ or $c$ by a red line and in either case $G_8$ would clearly contain two disjoint red triangles.

It follows, therefore, that $\mu(G_8)=2$ for any chromatic graph $G_8$.

To complete the proof of (2) we need only consider the possibility that $\mu(G_n)=[\frac{1}{3}n]-1$ for some graph $G_n$, where $n=3h+2$ and $h\geqq 3$. Let $T$ denote some maximal family of $\mu(G_n)=h-1$ mutually disjoint monochromatic triangles of $G_n$. There are $(3h+2)-3(h-1)=5$ points not involved in triangles of $T$. These five points and the three points belonging to any one triangle in $T$ determine a chromatic subgraph $G_8$ that, according to what we have already shown, contains two disjoint monochromatic triangles. These two triangles plus the remaining $h-2$ triangles in $T$ form a set of $h=[\frac{1}{3}n]$ mutually disjoint monochromatic triangles. This suffices to complete the proof of the theorem.

The reader should have little difficulty in constructing examples of graphs $G_n$ that show that inequality (1) is the best possible when $n\geqq 3$ and (2) doesn't apply.

Various extensions of this problem can be treated by similar methods; however, the resulting inequalities do not seem to be as sharp in general and we shall not pursue this further here.

**References**

1. P. Erdös and J. W. Moon, On subgraphs of the complete bipartite graph, Canad. Math. Bull., 7(1964) 35–39.

2. A. W. Goodman, On sets of acquaintances and strangers at any party, Amer. Math. Monthly, 66(1959) 778–783.

3. R. E. Greenwood and A. M. Gleason, Combinatorial relations and chromatic graphs, Canad. J. Math., 7(1955) 1–7.

4. G. Lorden, Blue-empty chromatic graphs, Amer. Math. Monthly, 69 (1962) 114–120.

5. J. W. Moon and L. Moser, On chromatic bipartite graphs, this MAGAZINE, 35 (1962) 225–227.

6. L. Sauvé, On chromatic graphs, Amer. Math. Monthly, 68 (1961) 107–111.

## APPROXIMATE TRISECTION OF AN ANGLE WITH EUCLIDEAN TOOLS

TZER-LIN CHEN, Taipei, Taiwan

I discovered the method by making an error in the solution of simultaneous equations in my paper: "Proof of the Impossibility of Trisecting an Angle with Euclidean Tools" (this MAGAZINE, vol. 39, pp. 239–241). I obtained the erroneous solution

$$\tan\frac{A}{3}=\frac{-1+[1+(4/3)(\tan^2 A)]^{1/2}}{2\tan A}.$$

Because the percentage of error was large on trial angles, 4/3 was replaced by 85/63 after several trials.

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$$\tan\frac{A}{3}=\frac{-1+[1+(4/3)(\tan^2 A)]^{1/2}}{2\tan A}.$$

Because the percentage of error was large on trial angles, 4/3 was replaced by 85/63 after several trials.

On a construction line $OQ$ (see Fig. 1) choose a short segment $l$ as a unit of measure. Use compasses to measure off $OJ=63l$ and $JK=85l$ on line $OQ$. Through $O$ draw any line $OP\neq OQ$ and on it mark off $OL=1$. Draw $LJ$ and construct $KY$ parallel to $LJ$; then $LY: OL=JK: OJ$ and $LY=85/63$.

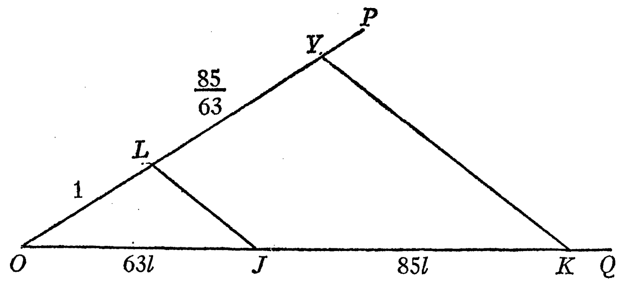


FIG. 1.

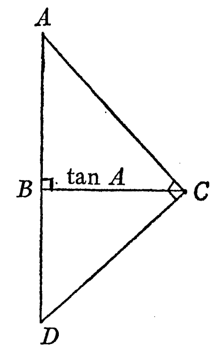


FIG. 2.

In Fig. 2, angle $A$ is the angle which is to be trisected. Triangle $ABC$ is a right triangle with right angle at $B$. Triangle $ACD$ is also a right triangle with right angle at $C$ and $AB=1$; $BC=\tan A$. $\triangle ABC$ is similar to $\triangle BCD$. Therefore $BD/BC=BC/AB$ so that $BD=\overline{BC}^2=\tan^2 A$.

In Fig. 3 construct $EF=OL=1$, $FG=LY=85/63$, and $EH=BD$ and draw $HF$. Construct a parallel to $HF$ from $G$ and let it intersect the extension of $EH$ at $I$. We have $HI:EH=FG:EF$ so that $HI=(85/63)\tan^2 A$.

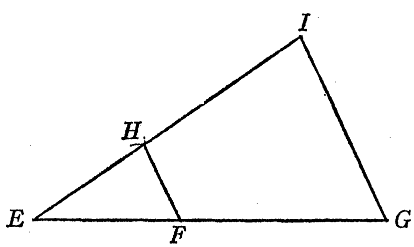


FIG. 3.

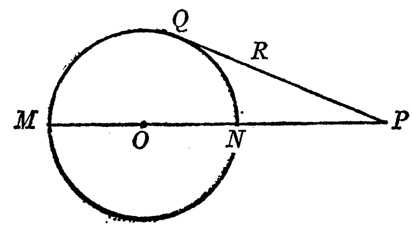


FIG. 4.

Construct a circle, $O$, whose radius is $HI/2$, and let $MN$ be a diameter of this circle. Prolong $MN$ to $P$ and make $NP=1$. Construct $PQ$ tangent to the circle $O$. (See Fig. 4.) Then $\overline{PQ}^2=PM\cdot PN$ and now mark off $PR=1$. Then

$$QR = -1 + \left[1 + \left(\frac{85}{63}\right)\tan^2 A\right]^{1/2}.$$

In Fig. 5 take $ST=2BC=2\tan A$ and

$$TU = QR = -1 + \left[1 + \left(\frac{85}{63}\right)\tan^2 A\right]^{1/2}.$$

Draw a straight line through $S$, let $SV=1$, and draw $VT$. Through $U$ draw a line parallel to $VT$ and intersecting $SV$ at $W$. Then $VW:SU=TU:ST$ and

$$VW=\frac{-1+\left[1+\left(\frac{85}{63}\right)\tan^2 A\right]^{1/2}}{2\tan A}.$$

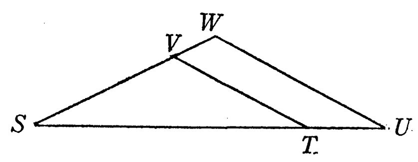


FIG. 5.

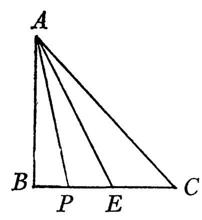


FIG. 6.

In the right triangle $ABC$ of Fig. 6 mark off on $BC$,

$$BP=\frac{-1+\left[1+\left(\frac{85}{63}\right)\tan^2 A\right]^{1/2}}{2\tan A}$$

and draw $AP$. Construct the bisector of angle $PAC$ and let it intersect $BC$ at $E$. Then $\angle PAE=\angle EAC$ and

$$\tan BAP=\frac{-1+\left[1+\left(\frac{85}{63}\right)\tan^2 A\right]^{1/2}}{2\tan A}.$$

Angles $PAE$ and $EAC$ are two approximate trisection angles of acute angle $A$. Therefore

$$\text{Approximate trisection angle}=\frac{\text{angle } A-\text{angle } BAP}{2}.$$

*Comments.* Suppose $e$ is the positive percentage of error in angle $BAP$. Then

$$\text{angle } BAP=\frac{\text{angle } A}{2}+e.$$

The approximate trisection angle

$$=\frac{\text{angle } A-\text{angle } BAP}{2}$$

and the approximate trisection angle

$$= \frac{\text{angle } A}{3} - \frac{e}{2}$$

therefore the negative percentage of error of approximate trisection angle is half of angle $BAP$.

We shall compare for one angle the value of the approximate trisection with the values given in trigonometry tables.

From tables, $\tan 33° = 0.6494$ and $\tan 11° = 0.1944$. Hence

$$\tan BAP = \frac{-1 + \left[1 + \frac{85}{63} (\tan 33°)^2\right]^{1/2}}{2 \tan 33°} = 0.1944$$

error $= 0$, percentage of error of $\tan BAP = 0\%$ (to 3 significant figures). Therefore the percentage of error in the approximate trisection angle 11° is 0% (to 3 significant figures).

We found when trisecting angles of 3°, 6°, 9°, $\cdots$, 81°, 84°, 87° into 1°, 2°, 3°, $\cdots$, 27°, 28°, 29°, the maximum percentage of error was 0.573% and the minimum was 0% (again, to 3 significant figures).

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## A NEW TYPE OF MEAN VALUE THEOREM

DONALD H. TRAHAN, U. S. Naval Postgraduate School

The first theorem of this article is a generalization of a theorem due to T. M. Flett [2]. Flett's theorem is Corollary 1 of this note, and this theorem is also given by R. P. Boas in his book on real analysis [1]. As compared to the Cauchy and Taylor mean value theorems, Flett's theorem seems to be a new type of mean value theorem. The purpose of this paper is to generalize Flett's result (this is accomplished by Theorems 1 and 2) and to indicate some of the numerous mean value theorems of this type.

Of course, mean value theorems can be used to derive inequalities and this is also the case for the theorems of this article. In fact, this is perhaps more apparent for the mean value theorems that are given here, since each theorem of this paper contains an inequality in the hypothesis. Some examples are given near the end of this paper in which inequalities are derived from theorems given here.

The hypotheses and conclusions of the theorems of this note have a variety of simple geometric interpretations. For example, the conclusion of the first theorem states that the tangent at $c$ passes through the initial point $(a, f(a))$.

We begin by stating two lemmas which give some simple and basic principles that are used throughout this article. It is interesting to compare these lemmas with Rolle's Theorem.

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We begin by stating two lemmas which give some simple and basic principles that are used throughout this article. It is interesting to compare these lemmas with Rolle's Theorem.

LEMMA 1. *If $f$ is continuous on $[a, b]$, $f$ is differentiable on $(a, b]$, and $f'(b)[f(b)-f(a)]\leqq 0$, then there exists $c\in(a, b]$ such that $f'(c)=0$.*

*Proof.* If $f(b)=f(a)$, then by Rolle's Theorem there exists $c\in(a, b)$ such that $f'(c)=0$. If $f'(b)=0$ then let $c=b$. If $f'(b)[f(b)-f(a)]<0$ then there is a maximum or minimum value at $c\in(a, b)$ and $f'(c)=0$.

LEMMA 2. *If $f$ is continuous on $[a, b]$, $f$ is differentiable on $(a, b]$, and $f'(b)[f(b)-f(a)]<0$, then there exists $c\in(a, b)$ such that $f'(c)=0$.*

THEOREM 1. *If $f$ is differentiable on $[a, b]$ and*

$$\left[f'(b) - \frac{f(b) - f(a)}{b - a}\right]\left[f'(a) - \frac{f(b) - f(a)}{b - a}\right] \geqq 0,$$

*then there exists $c\in(a, b]$ such that*

$$f'(c) = \frac{f(c) - f(a)}{c - a}.$$

COROLLARY 1. *If $f$ is differentiable on $[a, b]$ and $f'(a)=f'(b)$, then there exists $c\in(a, b)$ such that*

$$f'(c) = \frac{f(c) - f(a)}{c - a}.$$

COROLLARY 2. *If $f$ is differentiable on $[a, b]$, $f'(a)$ and $f'(b)$ are both less than or both greater than*

$$\frac{f(b) - f(a)}{b - a},$$

*then there exists $c\in(a, b)$ such that*

$$f'(c) = \frac{f(c) - f(a)}{c - a}.$$

*Proof of Theorem* 1. On $(a, b]$ let

$$h(x) = \frac{f(x) - f(a)}{x - a}$$

and let $h(a)=f'(a)$; then $h$ is continuous on $[a, b]$, $h$ is differentiable on $(a, b]$, and $(x-a)^2h'(x)=(x-a)f'(x)-[f(x)-f(a)]$. By Lemma 1, providing $h'(b)[h(b)-h(a)]\leqq 0$, there exists $c\in(a, b]$ such that $h'(c)=0$ or so that $(c-a)f'(c)=f(c)-f(a)$.

*Proof of Corollary* 1. Case 1: $f(b)-f(a)=f'(b)[b-a]$. Define $h$ as in the proof of Theorem 1; then $h(b)=h(a)$ and by Rolle's Theorem there exists $c\in(a, b)$ such that $h'(c)=0$. Case 2: $f(b)-f(a)\neq f'(b)[b-a]$. The result then follows by Theorem 1 and we note that $c\neq b$.

The proof of Corollary 2 is left to the reader.

THEOREM 2. *If $f$ and $g$ are differentiable on $[a, b]$, $g'(a)\neq 0$, $g(x)\neq g(a)$ for all $x\in(a, b]$, and*

$$\left[\frac{f'(a)}{g'(a)}-\frac{f(b)-f(a)}{g(b)-g(a)}\right]\Big[[g(b)-g(a)]f'(b)-[f(b)-f(a)]g'(b)\Big]\geqq 0,$$

*then there exists $c\in(a, b]$ such that $[g(c)-g(a)]f'(c)=[f(c)-f(a)]g'(c)$.*

COROLLARY 3. *If $f$ and $g$ are differentiable on $[a, b]$, $g'(a)\neq 0$, $g(x)\neq g(a)$ for all $x\in(a, b]$, $g'(b)[g(b)-g(a)]>0$, and*

$$\frac{f'(a)}{g'(a)}=\frac{f'(b)}{g'(b)},$$

*then there exists $c\in(a, b)$ such that $[g(c)-g(a)]f'(c)=[f(c)-f(a)]g'(c)$.*

*Proof of Theorem* 2. Let

$$h(x)=\frac{f(x)-f(a)}{g(x)-g(a)}$$

if $x\neq a$ and let

$$h(a)=\frac{f'(a)}{g'(a)};$$

then $h$ is continuous on $[a, b]$, $h$ is differentiable on $(a, b]$, and

$$[g(x)-g(a)]^2h'(x)=[g(x)-g(a)]f'(x)-[f(x)-f(a)]g'(x).$$

By Lemma 1 there exists $c\in(a,b]$ such that $h'(c)=0$ or so that $[g(c)-g(a)]f'(c)=[f(c)-f(a)]g'(c)$.

*Proof of Corollary* 3. If

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(b)}{g'(b)},$$

define $h$ as in the proof of Theorem 2; then $h(a)=h(b)$ and the result follows by Rolle's Theorem. Otherwise the result follows by Theorem 2.

There are many other mean value theorems of this type. The first two theorems of this paper can be compared with Cauchy's mean value theorem, and the next three theorems are comparable to Taylor's Theorem. From now on the details of the proofs are not given, since the proofs are similar to the proofs of Theorem 1 and 2. However, as an aid to the reader who wishes to check the results, the function $h$ that was used in two of the proofs is given. Only those theorems that follow from Lemma 1 are given. In each case, Lemma 2 gives a corresponding theorem where the strict inequality is used and $c\in(a, b)$.

The next theorem seems especially interesting in contrast to Taylor's Theorem, for the hypothesis of Theorem 3 requires that $f$ be differentiable on $[a, b]$ and that $f^{(n)}(a)$ exists while the hypothesis to Taylor's Theorem is that $f^{(n-1)}(x)$ exists on $[a, b]$ and $f^{(n)}(x)$ exists on $(a, b)$. Theorem 4 is a generalization of

Theorem 3 and Theorem 4 seems interesting to the author (again in contrast to Taylor's Theorem) just because it is so easy to establish.

The cases $n=1, 2$ of Theorem 3 have simple geometric interpretations. For example, the conclusion of Theorem 3 for the case $n=2$ is that there exists $c\in(a, b]$ such that $f'(c)-f'(a)=f''(a)[c-a]$.

THEOREM 3. *If $f$ is differentiable on $[a, b]$, $f^{(n)}(a)$ exists, and*

$$\left[f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}[b-a]^k\right]\left[f'(b) - \sum_{k=1}^{n} \frac{f^{(k)}(a)}{(k-1)!}[b-a]^{k-1}\right] \leqq 0,$$

*then there exists $c\in(a, b]$ such that*

$$f'(c) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{(k-1)!}[c-a]^{k-1}.$$

*Proof.* Let

$$h(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}[x-a]^k.$$

THEOREM 4. *If $f$, $g$ are differentiable on $[a, b]$, $f^{(n)}(a)$ exists, and*

$$\left[f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}[g(b)-g(a)]^k\right]\left[f'(b) - \sum_{k=1}^{n} \frac{f^{(k)}(a)}{(k-1)!}[g(b)-g(a)]^{k-1}\right] \leqq 0,$$

*then there exists $c\in(a, b]$ such that*

$$f'(c) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{(k-1)!}[g(c)-g(a)]^{k-1}.$$

THEOREM 5. *If $f$, $g$ are differentiable on $[a, b]$ and*

$$\left[g'(a)[f(b)-f(a)] - f'(a)[g(b)-g(a)]\right][g'(a)f'(b) - f'(a)g'(b)] \leqq 0,$$

*then there exists $c\in(a, b]$ such that $f'(c)g'(a)=f'(a)g'(c)$.*

*Proof.* Let

$$h(x) = [f(x)-f(a)]g'(a) - f'(a)[g(x)-g(a)].$$

The following examples illustrate how the theorems of this article can be used to derive inequalities. Let $f(x)=Ln(x)$ with $a>0$; then since the conclusion of Theorem 3 with $n=1$ does not hold, it follows that

$$Ln(b) < Ln(a) + \frac{1}{a}(b-a).$$

Similarly, Theorem 3 with $n=2$ implies

$$Ln(a) + \frac{1}{a}(b-a) - \frac{a^{-2}}{2}(b-a)^2 < Ln(b).$$

Most readers will recognize that these results follow by the Taylor series expansion of $Ln(x)$ at $x=a$, and that these relations are often obtained in textbooks by applying one of the various mean value theorems. What is intriguing is that Theorem 3 requires the existence of the second derivative at $x=a$, while the Cauchy and Taylor theorems require the existence of the second derivative on $(a, b)$.

As a final example, let $f(x)=\sin(x)$, $g(x)=\cos(x)$ and apply Theorem 2. Since the conclusion of Theorem 2 does not hold, it follows that if $\sin(a)\neq 0$ and $\cos(x)\neq\cos(a)$ for all $x\in(a, b]$ then

$$0<\cot(a)+\frac{\sin(b)-\sin(a)}{\cos(b)-\cos(a)} \quad\text{and}\quad \sin(a)[\cos(b)-\cos(a)]<0.$$

**References**

1. R. P. Boas, A primer of real functions, MAA, Carus Mathematical Monograph No. 13, (1960) 121–122.

2. T. M. Flett, A mean value theorem, Math. Gaz., 42 (1958) 38–39.

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# GEOMETRICAL INTERPRETATIONS OF THE INEQUALITIES BETWEEN THE ARITHMETIC, GEOMETRIC AND HARMONIC MEANS

O. SHISHA, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio

**1.** The purpose of this note is to collect together some proofs of inequalities between arithmetic, geometric, and harmonic means, based on geometrical theorems. The only proof given here which may be new is the one of Section **5**; the author has not seen it anywhere in the literature.

**2.** Let $a_1, a_2, \cdots, a_n, q_1, q_2, \cdots, q_n$ be positive numbers with $\sum_{j=1}^n q_j=1$. We consider the arithmetic, geometric, and harmonic means given, respectively, by

$$A=\sum_{j=1}^n q_j a_j,\quad G=\prod_{j=1}^n a_j^{q_j},\quad\text{and}\quad H=\left[\sum_{j=1}^n q_j/a_j\right]^{-1}.$$

A classical theorem states that $H<G<A$ unless $a_1=a_2=\cdots=a_n$, in which case clearly $A=G=H=a_1$. An important special case is that in which $q_j=n^{-1}$ for $j=1, 2, \cdots, n$. In this case we have

$$A=\left[\sum_{j=1}^n a_j\right]\Big/ n,\quad G=\left[\prod_{j=1}^n a_j\right]^{1/n},\quad\text{and}\quad H=n\Big/\sum_{j=1}^n a_j^{-1}.$$

Except for the last section, we shall restrict ourselves to this special case.

**3.** Suppose $n=2$, $a_1\neq a_2$. Consider a right triangle, the projections of whose legs on its hypotenuse being $a_1$ and $a_2$. Then the altitude to the hypotenuse of the triangle is $G$, the length of the median to the hypotenuse is $A$ (prove!), and consequently $G<A$.

Most readers will recognize that these results follow by the Taylor series expansion of $Ln(x)$ at $x=a$, and that these relations are often obtained in textbooks by applying one of the various mean value theorems. What is intriguing is that Theorem 3 requires the existence of the second derivative at $x=a$, while the Cauchy and Taylor theorems require the existence of the second derivative on $(a, b)$.

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**4.** Suppose now $n=3$, and $a_1$, $a_2$, $a_3$ are not all equal. Set $p=a_1+a_2+a_3$, $y_j=p-a_j$ $(j=1, 2, 3)$. Then the inequality $G<A$ is

$$[(p-y_1)(p-y_2)(p-y_3)]^{1/3} < p/3,$$

which is equivalent to

$$(1) \qquad [p(p-y_1)(p-y_2)(p-y_3)]^{1/2} < \left[p\left(p-\frac{2p}{3}\right)^3\right]^{1/2}.$$

Let $T$ be a triangle (whose existence is easily seen) with sides of lengths $y_1$, $y_2$, $y_3$. Let $E$ be an equilateral triangle having the same perimeter $2p$ as $T$. Since $T$ is not equilateral, its area is (by a known theorem) smaller than the area of $E$. This last fact is, however, just the inequality (1), in view of Heron's formula for the area of a triangle.

**5.** We now let both $n$ and the positive $q_j$ (with $\sum_{j=1}^n q_j=1$) be arbitrary, and consider the inequality

$$(2) \qquad H = \left(\sum_{j=1}^{n} q_j/a_j\right)^{-1} \leqq \sum_{j=1}^{n} q_j a_j = A.$$

In $E_n$ consider the hyperplane $\alpha$:

$$\sum_{j=1}^{n} (q_j/a_j)^{1/2} x_j = 1.$$

Let $P$ be the foot of the perpendicular from the origin $O$ of $E_n$ to $\alpha$, and let $Q=((q_1a_1)^{\frac{1}{2}}, (q_2a_2)^{\frac{1}{2}}, \cdots, (q_na_n)^{\frac{1}{2}})$. Then $Q$ lies on $\alpha$ and therefore (with $\|\ \|$ denoting length)

$$(3) \qquad \left(\sum_{j=1}^{n} q_j/a_j\right)^{-1} = \|\overrightarrow{PO}\|^2 \leqq \|\overrightarrow{QO}\|^2 = \sum_{j=1}^{n} q_j a_j,$$

so we have obtained (2). Furthermore, if the $a_j$ are not all equal, then $\overrightarrow{QO}$ is not perpendicular to $\alpha$, so a strict inequality holds in (3), and hence $H<A$.

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## COMBINATIONS, SUCCESSIONS AND THE $n$-KINGS PROBLEM

MORTON ABRAMSON and WILLIAM MOSER, McGill University

Let $A=(a_{ij})$ be an $n\times m$ matrix with $nm$ distinct entries. Let $g_{n,k}(m)$ denote the number of ways of choosing $k$ of the $nm$ entries such that no two come from the same row, and any two from adjacent rows come from the same column. The main purpose of this note is to find $g_{n,k}(m)$ by elementary methods. The numbers $g_{n-1,k}(2)$, $k=0, 1, 2, \cdots$, are used in obtaining a solution of the "$n$-kings problem": in how many ways may $n$ kings be placed on an $n\times n$ chessboard, one in each row and column, so that no two attack each other? The sub-

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which is equivalent to

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Let $T$ be a triangle (whose existence is easily seen) with sides of lengths $y_1$, $y_2$, $y_3$. Let $E$ be an equilateral triangle having the same perimeter $2p$ as $T$. Since $T$ is not equilateral, its area is (by a known theorem) smaller than the area of $E$. This last fact is, however, just the inequality (1), in view of Heron's formula for the area of a triangle.

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$$H = \left(\sum_{j=1}^{n} q_j/a_j\right)^{-1} \leqq \sum_{j=1}^{n} q_ja_j = A. \tag{2}$$

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Let $P$ be the foot of the perpendicular from the origin $O$ of $E_n$ to $\alpha$, and let $Q=((q_1a_1)^{\frac{1}{2}}, (q_2a_2)^{\frac{1}{2}}, \cdots, (q_na_n)^{\frac{1}{2}})$. Then $Q$ lies on $\alpha$ and therefore (with $\|\ \|$ denoting length)

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so we have obtained (2). Furthermore, if the $a_j$ are not all equal, then $\overrightarrow{QO}$ is not perpendicular to $\alpha$, so a strict inequality holds in (3), and hence $H<A$.

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We call $k$ integers

$$x_1 < x_2 < \cdots < x_k, \tag{1}$$

chosen from $\{1, 2, \cdots, n\}$ a *k-choice* (combination, selection) *of n*. A *part* of (1) is a sequence of consecutive integers not contained in a longer sequence of consecutive integers. For example

$$1, 3, 4, 5, 8, 9 \tag{2}$$

is a 6-choice of 10, with parts (1), (3, 4, 5), (8, 9) of lengths 1, 3, 2 respectively.

The number of $k$-choices of $n$ with exactly $r$ parts is

$$g_r(n, k) = \binom{k-1}{r-1}\binom{n-k+1}{r}. \tag{3}$$

To establish this let a dot represent an element not chosen and a dash represent an element chosen. There is an obvious 1-1 correspondence between $k$-choices of $n$ with $r$ parts and ordered (in rising order from left to right) arrangements of $n-k$ dots and $k$ dashes, along a straight line, with exactly $r$ groups of consecutive dashes. To find all such arrangements, array the $n-k$ dots, forming $n-k+1$ cells (including the cell before the first dot and the cell after the last dot). We may choose $r$ of the cells in $\binom{n-k+1}{r}$ ways, and then distribute the $k$ dashes into them, with none of the $r$ cells empty, in $\binom{k-1}{r-1}$ ways, establishing (3).

In the case $r=k$ we have Kaplansky's Lemma [1]: The number of $k$-choices of $n$, no two consecutive, is

$$g_k(n, k) = \binom{n-k+1}{k}. \tag{4}$$

In a $k$-choice (1), a *succession* is a pair $x_i$, $x_{i+1}$ with $x_{i+1}-x_i=1$. A part of (1) of length $a$ gives rise to $a-1$ successions. Hence, if (1) has $r$ parts of lengths $a_1, a_2, \cdots, a_r$, then it has $s=a_1+a_2+\cdots+a_r-r=k-r$ successions. Putting $r=k-s$ in (3) gives Riordan's theorem [4]:

$$f_s(n, k) = g_{k-s}(n, k) = \binom{k-1}{s}\binom{n-k+1}{k-s},$$

the number of $k$-choices of $n$ containing exactly $s$ successions.

Now we consider similar results for integers arranged in a circle, where 1 and $n$ are considered to be consecutive. The number of circular $k$-choices of $n$ with exactly $r$ parts is

$$h_r(n, k) = \frac{n}{n-k}\binom{n-k}{r}\binom{k-1}{r-1}. \tag{5}$$

For, there is an obvious 1-1 correspondence between circular $k$-choices of $n$ with $r$ parts and cyclic arrangements of $n-k$ dots and $k$ dashes, with exactly $r$ groups of consecutive dashes, in which one of the $n$ dots and dashes is marked 1 (rising order being clockwise). To find all such arrangements, array $n-k$ dots in a circle forming $n-k$ cells. We may choose $r$ of the cells in

$$\frac{1}{n-k}\binom{n-k}{r}$$

distinct ways (for the $\binom{n-k}{r}$ choices fall into sets of $n-k$ each which are the same by rotation). Now we may distribute the $k$ dashes into the $r$ cells, with none empty, in $\binom{k-1}{r-1}$ ways, and then mark any one of the $n$ dots and dashes. Hence (5).

In the case $r=k$ we have Kaplansky's Lemma [1]: the number of circular $k$-choices of $n$, no two consecutive, is

$$h_k(n, k) = \frac{n}{n-k}\binom{n-k}{k}. \tag{6}$$

Putting $r=k-s$ in (5) gives

$$h_{k-s}(n, k) = \frac{n}{n-k}\binom{n-k}{k-s}\binom{k-1}{s},$$

the number of circular $k$-choices of $n$ containing exactly $s$ successions.

The recurrence

$$g_r(n, k) = g_r(n-1, k) + g_{r-1}(n-2, k-1) + g_r(n-1, k-1) - g_r(n-2, k-1) \tag{7}$$

can be established as follows. If a $k$-choice of $n$ with $r$ parts:

(i) does not contain $n$, then it is a $k$-choice of $n-1$ with $r$ parts, and there are $g_r(n-1, k)$ of these;

(ii) contains $n$ but not $n-1$, then deleting the $n$ we obtain a $(k-1)$-choice of $n-2$ with $r-1$ parts, and there are $g_{r-1}(n-2, k-1)$ of these;

(iii) contains $n$ and $n-1$, then deleting the $n$ we obtain a $(k-1)$-choice of $n-1$, with $r$ parts, containing $n-1$, and there are $g_r(n-1, k-1)-g_r(n-2, k-1)$ of these, since, of the $g_r(n-1, k-1)$ $(k-1)$-choices of $n-1$ with $r$ parts there are $g_r(n-2, k-1)$ not containing $n-1$.

In terms of Riordan's $f_s(n, k)$, (7) is the recurrence of the lemma in [4].

The recurrence (7) is equivalent to

$$g_r(n, k) = g_r(n-1, k) + \sum_{j=1}^{k-1} g_{r-1}(n-j-1, k-j), \qquad 2 \leqq r \leqq k, \tag{8}$$

while $g_1(n, k)=g_1(n-1, k)+1$. Of course $g_1(n, k)=n-k+1$. Using induction, we may deduce (3) from (8).

We now determine the numbers $g_{n,k}(m)$ described in the introduction. A $k$-choice of the $n$ rows (of $A$), with exactly $r$ parts, gives rise to $m^r$ choices counted in $g_{n,k}(m)$. Hence

$$g_{n,k}(m) = \sum_{r=1}^{k} m^r g_r(n, k) = \sum_{r=1}^{k} m^r \binom{k-1}{r-1}\binom{n-k+1}{r}, \tag{9}$$

by the use of (3).

Also, we note that $g_{n,k}(m)$ is the coefficient of $x^k$ in

$$\begin{aligned}(1 + mx + mx^2 + mx^3 + \cdots)^{n-k+1} \\ &= [1 + (m-1)x]^{n-k+1}(1-x)^{-(n-k+1)} \\ &= \sum_{i=0} \binom{n-k+1}{i}(m-1)^i x^i \sum_{j=0} \binom{n-k+j}{j} x^j.\end{aligned}$$

Putting $j = k - i$, we see that the coefficient of $x^k$ is

$$g_{n,k}(m) = \sum_{i=0}^{k} (m-1)^i \binom{n-i}{k-i}\binom{n-k+1}{i}. \tag{10}$$

This expression may also be deduced from (9) since

$$\begin{aligned}\sum_{r=1}^{k} m^r \binom{k-1}{r-1}\binom{n-k+1}{r} &= \sum_{r=0}^{k}\binom{k-1}{k-r}\binom{n-k+1}{r}\sum_{s=0}^{r}\binom{r}{s}(m-1)^s \\ &= \sum_{s=0}^{k}(m-1)^s \sum_{r=s}^{k}\binom{r}{s}\binom{n-k+1}{r}\binom{k-1}{k-r} \\ &= \sum_{s=0}^{k}(m-1)^s\binom{n-k+1}{s}\sum_{r=s}^{k}\binom{n-k+1-s}{r-s}\binom{k-1}{k-r} \\ &= \sum_{s=0}^{k}(m-1)^s\binom{n-k+1}{s}\binom{n-s}{k-s}.\end{aligned}$$

From (7) and (9) we have the recurrence

$$g_{n,k}(m) = g_{n-1,k}(m) + g_{n-1,k-1}(m) + (m-1)g_{n-2,k-1}(m). \tag{11}$$

Since $g_{n,k}(1) = \binom{n}{k}$, (11) in the case $m = 1$ is simply

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The numbers $A_{n,k}$ of Kaplansky ([2], [3]) are our

$$g_{n-1,k}(2) = \sum_{i=0}^{k}\binom{n-i-1}{k-i}\binom{n-k}{i} = \sum_{i=0}^{k}\binom{n-k+i-1}{i}\binom{n-k}{k-i},$$

in agreement with Riordan [5, p. 710]. They are used in solving the "$n$-kings problem" as follows. Suppose the king in column $i$ is in row $\pi(i)$. Since no two kings are in a row and no two in a column, $\pi(1), \pi(2), \cdots, \pi(n)$ is a permutation of $1, 2, \cdots, n$. Since no two kings attack each other

$$|\pi(s+1) - \pi(s)| \neq 1, \qquad s = 1, 2, \cdots, n-1.$$

Thus, if we let $(i, j)$ denote the event "$i$ immediately precedes $j$" (in an arbitrary permutation of $1, 2, \cdots$), a permutation corresponding to a permissible positioning of the $n$ kings is one containing none of the $2n-2$ events displayed in the following $n-1\times 2$ array:

$$(12)\qquad \begin{array}{cc} (1,2) & (2,1) \\ (2,3) & (3,2) \\ \vdots & \vdots \\ (n-1,n) & (n,n-1) \end{array}$$

Clearly, a permutation cannot contain two events from the same row of (12), and if it contains two events from adjacent rows of (12) they must come from the same column. Hence, the number of ways of choosing $k$ consistent events from (12) is $g_{n-1,k}(2)$. Furthermore, it is not difficult to see that the number of permutations containing a particular choice of $k$ consistent events is $(n-k)!$. Hence, by the well-known Principle of Inclusion and Exclusion, the solution to the $n$ kings problem is

$$\sum_{k=0}^{n-1}(-1)^k g_{n-1,k}(2)(n-k)!$$

with $g_{n-1,0}(2)=1$.

**References**

1. I. Kaplansky, Solution of the "problème des ménages," Bull. Amer. Math. Soc., 49 (1943) 784–5.

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# A GEOMETRIC INTERPRETATION OF THE SOLUTIONS OF $y''=c^2y$

D. A. LIND, University of Virginia

The general solution of the equation $y''=c^2y$, where $c^2\neq 0$ is a real constant, may be written in the form $y=k_1f(x)+k_2g(x)$, where $f(x)=(e^{cx}-e^{-cx})/2c$, $g(x)=(e^{cx}+e^{-cx})/2$, and where $k_1$ and $k_2$ are arbitrary constants. We note that for the special case $c^2=-1$ we have $f(x)=\sin x$, $g(x)=\cos x$, while for $c^2=1$ we find $f(x)=\sinh x$, $g(x)=\cosh x$. It is our aim to show that for any choice of real nonzero $c^2$ we can associate $f(x)$ and $g(x)$ with functions analogous to trigonometric and hyperbolic functions, but defined in terms of a conic section of eccentricity $\epsilon=(1+1/c^2)^{\frac{1}{2}}$. We shall also indicate that these generalized functions extend many properties common to trigonometric and hyperbolic functions.

positioning of the $n$ kings is one containing none of the $2n-2$ events displayed in the following $n-1\times 2$ array:

$$(12)\qquad \begin{array}{cc} (1,2) & (2,1) \\ (2,3) & (3,2) \\ \vdots & \vdots \\ (n-1,n) & (n,n-1) \end{array}$$

Clearly, a permutation cannot contain two events from the same row of (12), and if it contains two events from adjacent rows of (12) they must come from the same column. Hence, the number of ways of choosing $k$ consistent events from (12) is $g_{n-1,k}(2)$. Furthermore, it is not difficult to see that the number of permutations containing a particular choice of $k$ consistent events is $(n-k)!$. Hence, by the well-known Principle of Inclusion and Exclusion, the solution to the $n$ kings problem is

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# A GEOMETRIC INTERPRETATION OF THE SOLUTIONS OF $y''=c^2y$

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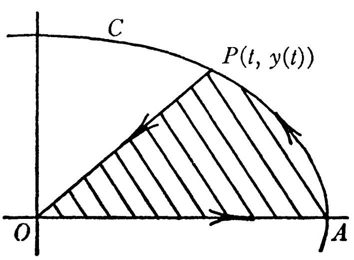
The general solution of the equation $y''=c^2y$, where $c^2\neq 0$ is a real constant, may be written in the form $y=k_1f(x)+k_2g(x)$, where $f(x)=(e^{cx}-e^{-cx})/2c$, $g(x)=(e^{cx}+e^{-cx})/2$, and where $k_1$ and $k_2$ are arbitrary constants. We note that for the special case $c^2=-1$ we have $f(x)=\sin x$, $g(x)=\cos x$, while for $c^2=1$ we find $f(x)=\sinh x$, $g(x)=\cosh x$. It is our aim to show that for any choice of real nonzero $c^2$ we can associate $f(x)$ and $g(x)$ with functions analogous to trigonometric and hyperbolic functions, but defined in terms of a conic section of eccentricity $\epsilon=(1+1/c^2)^{\frac{1}{2}}$. We shall also indicate that these generalized functions extend many properties common to trigonometric and hyperbolic functions.

For the rest of this note we neglect the parabola. Let $C$ be a conic section given by

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-\epsilon^2)}=1, \tag{1}$$

where we allow $\epsilon\in[0,\infty)-\{1\}$ or $\epsilon=\alpha i$ for $\alpha\in[0,\infty)$ and $i^2=-1$. In the first case, we have an ellipse with its major axis on the $x$-axis for $0\leqq\epsilon<1$ and a hyperbola for $1<\epsilon<\infty$, and in the latter case the conic section is an ellipse with its major axis on the $y$-axis. With $O$ the center of $C$, let $A$ be the point of intersection of $C$ with the $+x$-axis, and $P(t, y(t))$ be a point of $C$. Define $C(t)$ to be the closed curve determined by the arc of $C$ from $A$ to $P$ in a positive direction along with the directed segments $\overrightarrow{OA}$ and $\overrightarrow{PO}$, as suggested in the figure. The conic




angle $u$ between $\overrightarrow{OA}$ and $\overrightarrow{PO}$ is defined to be twice the shaded area inside $C(t)$ divided by $a^2$, so that

$$u=\frac{2}{a^2}\int_{C(t)}\tfrac{1}{2}(xdy-ydx). \tag{2}$$

For the circle ($\epsilon=0$) $u$ is the radian measure of angle $POA$, while for the equilateral hyperbola ($\epsilon=\sqrt{2}$) $u$ is the measure of "hyperbolic angle" discussed by Shervatov in his monograph on hyperbolic functions [1]. Finally, we define the generalized conic functions of $u$ based on a conic section of eccentricity $\epsilon$ by

$$\begin{aligned} \operatorname{sinc}_\epsilon u &= y/a & \qquad \operatorname{ctnc}_\epsilon u &= t/y\\ \operatorname{cosc}_\epsilon u &= t/a & \qquad \operatorname{secc}_\epsilon u &= a/t\\ \operatorname{tanc}_\epsilon u &= y/t & \qquad \operatorname{cscc}_\epsilon u &= a/y \end{aligned}$$

where $y=y(t)$. For $\epsilon=0$ these reduce to the trigonometric functions, and for $\epsilon=\sqrt{2}$ they become the hyperbolic functions.

We shall now show that $\operatorname{sinc}_\epsilon x$ and $\operatorname{cosc}_\epsilon x$ are two linearly independent solutions of $y''=c^2y$, where $\epsilon=(1+1/c^2)^{\frac{1}{2}}$, by showing they have the required exponential forms mentioned previously. The parametric representation of the ellipse is $x=a\cos v$, $y=b\sin v$. Letting $t=a\cos v_0$, we have from (2) that

$$u=\frac{1}{a^2}\int_0^{v_0}(ab\cos^2 v+ab\sin^2 v)dv=\frac{b}{a}v_0.$$

In order to get $u$ in terms of $t, y=b \sin v_0$, and $\epsilon$, we let $w=(\epsilon^2-1)^{-\frac{1}{2}}=-ia/b$, so that

$$u = (1/w) \ln (e^{-iv_0}) = (1/w) \ln \left(\frac{t + yw}{a}\right).$$

Similarly, the parametric representation of the hyperbola is $x=a \cosh v$, $y=b \sinh v$, so that if $t=a \cosh v_0$,

$$u = \frac{1}{a^2}\int_0^{v_0} (ab \cosh^2 v - ab \sinh^2 v)dv = \frac{b}{a} v_0.$$

Again to find $u$ as a function of $t$, $y=b \sinh v_0$, and $\epsilon$, we let $w=(\epsilon^2-1)^{-\frac{1}{2}}=a/b$, finding that

$$u = (1/w) \ln e^{v_0} = (1/w) \ln \left(\frac{t + yw}{a}\right).$$

Thus $u$ has the same form for all conic sections, and we have the relation

$$e^{uw} = \frac{t + vw}{a}\,. \tag{3}$$

Now it is easy to show using (1) that

$$(1/2w)\left(\frac{t + yw}{a} - \frac{a}{t + yw}\right) = y/a, \qquad \frac{1}{2}\left(\frac{t + yw}{a} + \frac{a}{t + yw}\right) = t/a,$$

so that with (3) we obtain

$$\operatorname{sinc}_\epsilon u = \frac{e^{uw} - e^{-uw}}{2w}, \qquad \operatorname{cosc}_\epsilon u = \frac{e^{uw} + e^{-uw}}{2}\,. \tag{4}$$

The exponential forms for the other conic functions follow immediately. Comparison with the general solution of $y''=c^2y$ given above shows that it may now be written as

$$y = k_1 \operatorname{sinc}_\epsilon x + k_2 \operatorname{cosc}_\epsilon x,$$

where $\epsilon=(1+1/c^2)^{\frac{1}{2}}$.

We note that the conic functions are related to trigonometric and hyperbolic functions by the following equations, in which $w=(\epsilon^2-1)^{-\frac{1}{2}}$.

$$\operatorname{sinc}_\epsilon u = \frac{i}{w} \sin \left(\frac{uw}{i}\right) = \frac{i}{w} \sinh (uw), \tag{5}$$

$$\operatorname{cosc}_\epsilon u = \cos \left(\frac{uw}{i}\right) = \cosh (uw). \tag{6}$$

It is a consequence of the exponential forms (4) that many properties of trigonometric and hyperbolic functions generalize directly. The following representative identities may be verified by substitution or by using (5) and (6).

$$\operatorname{cosc}_\epsilon^2 u - w^2 \operatorname{sinc}_\epsilon^2 u = 1$$

$$\operatorname{sinc}_\epsilon (u \pm v) = \operatorname{sinc}_\epsilon u \operatorname{cosc}_\epsilon v \pm \operatorname{sinc}_\epsilon v \operatorname{cosc}_\epsilon u$$

$$\operatorname{cosc}_\epsilon (u \pm v) = \operatorname{cosc}_\epsilon u \operatorname{cosc}_\epsilon v \pm w^2 \operatorname{sinc}_\epsilon u \operatorname{sinc}_\epsilon v$$

$$(\operatorname{cosc}_\epsilon u \pm w \operatorname{sinc}_\epsilon u)^n = \operatorname{cosc}_\epsilon nu \pm w \operatorname{sinc}_\epsilon nu \qquad (n = 0, 1, \cdots)$$

This work was supported by the Undergraduate Research Participation Program of the National Science Foundation through GE 8026.

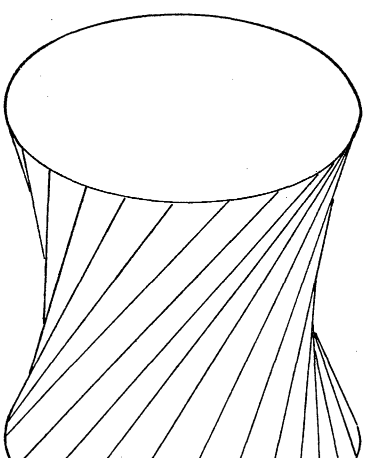
**Reference**

1. V. G. Shervatov, Hyperbolic Functions, D. C. Heath, Boston, 1963.

## GENERATING A HYPERBOLOID

C. STANLEY OGILVY, Hamilton College

Almost anyone is astonished when he first learns that a hyperboloid of revolution contains straight lines, or *rulings*, in the surface. Such a surface can be generated (swept out) by rotating a straight line about an axis skew to it (Fig. 1). Most of the elementary texts don't prove this, making instead the usual excuse that advanced methods are necessary. Even Hilbert and Cohn-Vossen, who seem to be able to do almost anything with visual methods, state "The proof of this fact can only be given analytically" [1]. It might therefore be advantageous to have on tap a short and easy proof.



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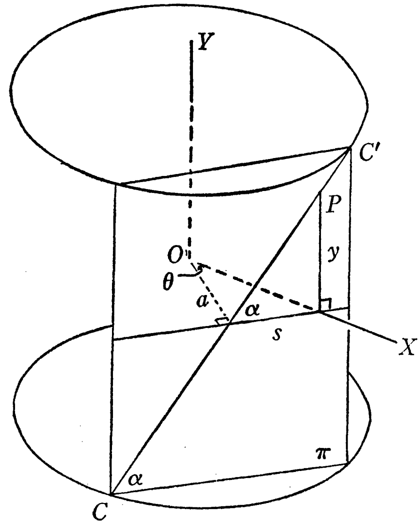


FIG. 2.

Let $CC'$, the generating line that is being rotated around the $Y$ axis, cut the $XY$-plane at $P$ (Fig. 2). We seek the coordinates $(x, y)$ of $P$. In the vertical

$$\operatorname{cosc}_\epsilon^2 u - w^2 \operatorname{sinc}_\epsilon^2 u = 1$$

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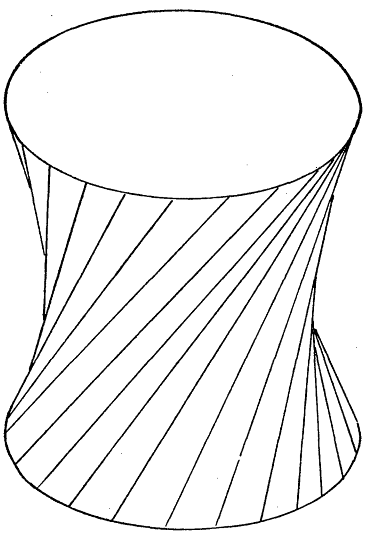

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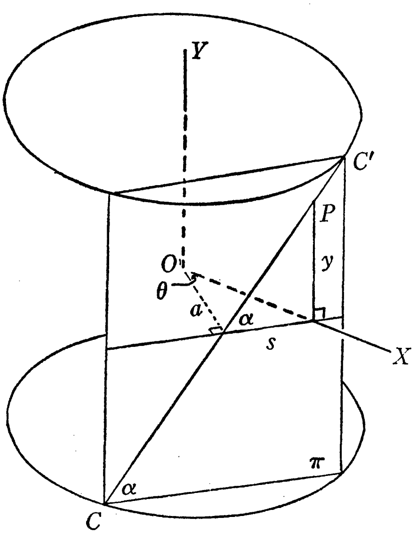


FIG. 2.

Let $CC'$, the generating line that is being rotated around the $Y$ axis, cut the $XY$-plane at $P$ (Fig. 2). We seek the coordinates $(x, y)$ of $P$. In the vertical plane $\pi$ containing $CC'$, $\tan \alpha = y/s$. Let the distance from $O$ to $CC'$ be called $a$, a suitable choice because when the plane $\pi$ is perpendicular to $OX$, $a$ will be the $x$-coordinate of $P$. In the horizontal plane through $O$, $\tan \theta = s/a$. Eliminating $s$ between the last two equations, we have $y/(a \tan \alpha) = \tan \theta$. But $x/a = \sec \theta$. Now elimination of $\theta$, the parameter of rotation, will yield the locus of $P$ for various positions of the line $CC'$. Squaring and subtracting, we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $b = a \tan \alpha$, or $\tan \alpha = b/a$, the slope of an asymptote.

We have found the equation of the hyperbola traced out by the point $P$ on the $XY$-plane as $CC'$ is rotated around $OY$. The same could be done in any vertical plane, and the surface is therefore a hyperboloid of revolution. The line $CC'$ cuts the $XY$-plane for all $\theta$ except $\theta = \pi/2$ (and $3\pi/2$), and hence we could have predicted that $\tan \alpha$ would be the slope of an asymptote.

**Reference**

1. *D. Hilbert & S. Cohn-Vossen, Geometry and the Imagination, Chelsea, New York, 1952.*

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# PARAMETRIZATION OF CERTAIN QUADRICS

A. R. AMIR-MOÉZ, Texas Technological College

A parametric set of equations for an equation in several variables may be useful in calculating definite integrals. In this note we express the equation

$$\sum_{i=1}^{n} \left(\frac{x_i}{a_i}\right)^2 = 1,$$

where $x_i$ is a real variable and $a_i$ is a real number for all $i$, in a parametric form. Then we apply the results to integration.

**1. An inductive approach.** It is clear that the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

can be written as

$$\begin{cases} x = a \cos t \\ y = b \sin t. \end{cases}$$

Now we shall go one step ahead. Let us consider

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

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A parametric form of this ellipsoid is

$$\begin{cases} x = a \cos t \\ y = b \sin t \cos s \\ z = c \sin t \sin s. \end{cases}$$

This suggests that for the general case we can write

$$(1) \qquad \begin{cases} x_1 = a_1 \cos t_1 \\ x_2 = a_2 \sin t_1 \cos t_2 \\ \cdots \qquad \cdots \qquad \cdots \\ x_{n-1} = a_{n-1} \sin t_1 \cdots \sin t_{n-2} \cos t_{n-1} \\ x_n = a_n \quad \sin t_1 \cdots \sin t_{n-1}. \end{cases}$$

It is easy to show that

$$\sum_{i=1}^{n} \left(\frac{x_i}{a_i}\right)^2 = 1.$$

**2. An application to integration.** Let us, as an example, obtain the volume of an ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

This problem is considered as a very tedious one. We shall use the parametric representation described in (1). The volume is, [1],

$$V = \left| \int\int_R xdydz \right|,$$

where $R$ is the proper region for the integration. Let

$$\begin{cases} x = a \cos t \\ y = b \sin t \cos s \\ z = c \sin t \sin s. \end{cases}$$

We compute the Jacobian

$$\frac{\partial(y, z)}{\partial(t, s)} = \begin{vmatrix} b \cos t \cos s & c \cos t \sin s \\ -b \sin t \sin s & c \sin t \cos s \end{vmatrix} = bc \sin t \cos t.$$

Thus

$$V = 8abc \left| \int_0^{\pi/2} ds \int_0^{\pi/2} \sin t \cos^2 t dt \right| = \frac{4}{3} \pi abc.$$

**3. The Jacobian matrix in the general case.** In Section 2 the Jacobian matrix is

$$\begin{pmatrix} b\cos t\cos s & c\cos t\sin s \\ -b\sin t\sin s & c\sin t\cos s \end{pmatrix}.$$

This matrix can be written as

$$bc\sin t\cos t\begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} = bc\sin t\cos t\,(A).$$

We observe that the matrix $(A)$ is unitary. This suggests the test for the general case. In fact for

$$\sum_{i=1}^{n}\left(\frac{x_i}{a_i}\right)^2 = 1$$

and the change of variable (1), the Jacobian matrix will be

$$\left(\prod_{i=2}^{n} a_i\right)(\cos t_1 \sin^{n-2} t_1 \cdots \sin t_{n-2})(A),$$

where the matrix $(A)$ is unitary. We leave the verification to the reader. Thus

$$|\det(A)| = 1,$$

where $\det(A)$ means the determinant of the matrix $(A)$. Therefore

$$\frac{\partial(x_2, \cdots, x_n)}{\partial(t_1, \cdots, t_{n-1})} = \left(\prod_{i=2}^{n} a_i\right)\cos t_1 \sin^{n-2} t_1 \cdots \sin t_{n-2} \det(A).$$

**4. The volume of a hyperellipsoid.** Consider the $(n-1)$st order integral

$$V = \left|\int_R \cdots \int x_1 dx_2 \cdots dx_n\right|$$

for real variables $x_1, \cdots, x_n$ satisfying

$$\sum_{i=1}^{n}\left(\frac{x_i}{a_i}\right)^2 = 1.$$

Here we can write

$$V = 2^n\left(\prod_{i=1}^{n} a_i\right)\left|\int_0^{\pi/2} dt_{n-1}\int_0^{\pi/2}\cos^2 t_1 \sin^{n-2} t_1 dt_1 \int_0^{\pi/2}\sin^{n-3} t_2 dt_2 \cdots \int_0^{\pi/2}\sin t_{n-2} dt_{n-2}\right|.$$

With the use of $\Gamma$ functions [2] this integral can be simplified to

$$V = \frac{(\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}+1\right)}\left(\prod_{i=1}^{n} a_i\right),$$

where $n$ is a positive integer denoting the dimension of the space.

**5. Parametrization of hyperboloids.** Consider the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

A set of parametric equations for this hyperbola is

$$\begin{cases} x = a\cosh t \\ y = b\sinh t. \end{cases}$$

Let us consider the next simplest case. For

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

we can write

$$\begin{cases} x = a\cosh t \\ y = b\sinh t\cosh s \\ z = c\sinh t\sinh s. \end{cases}$$

In fact for

$$\left(\frac{x_1}{a_1}\right)^2-\left(\frac{x_2}{a_2}\right)^2+\cdots+\left(\frac{x_{2k-1}}{a_{2k-1}}\right)^2-\left(\frac{x_{2k}}{a_{2k}}\right)^2+\cdots\pm\left(\frac{x_n}{a_n}\right)^2=1 \tag{2}$$

this idea can be generalized, and we can write

$$\begin{cases} x_1 = a_1\cosh t_1 \\ x_2 = a_2\sinh t_1\cosh t_2 \\ \cdots \qquad \cdots \\ x_{n-1} = a_{n-1}\sinh t_1\cdots\sinh t_{n-2}\cosh t_{n-1} \\ x_n = a_n\sinh t_1\cdots\sinh t_{n-2}\sinh t_{n-1}. \end{cases} \tag{3}$$

We leave the verification to the reader.

Now if we have

$$\sum_{i=1}^{p}\left(\frac{x_i}{a_i}\right)^2-\sum_{i=p}^{n}\left(\frac{x_i}{a_i}\right)^2=1$$

which is not of the same form as (2), then some modification of (3) can be obtained.

**6. Use of imaginary numbers.** Again we consider the $n$-dimensional hyperboloid of Section 5. Then a set of parametric equations for this hyperboloid may be obtained from (1) using $i=\sqrt{-1}$. We leave this and its applications to the reader.

**References**

1. T. M. Apostol, Calculus, Blaisdell, New York, 1962, p. 78.
2. A. E. Taylor, Advanced Calculus, Ginn & Co., Boston, 1955, p. 648.

# INTEGERS THAT ARE MULTIPLIED WHEN THEIR DIGITS ARE REVERSED

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**1. Introduction.** A good deal of attention has been paid to numbers that are multiplied by an integer when a cyclic shift of their digits is made. The purpose of this note is to consider numbers that give a multiple of themselves when their digits are reversed. For example, in the decimal scale, the only such numbers less than 10000 are $9\times 1089=9801$ and $4\times 2178=8712$. The cases of 2-digit and 3-digit numbers, in any scale of notation, are considered fully here while a partial solution is given for the 4-digit case. There is also a discussion of the nature of parametric solutions.

In general, we are interested in the solution of

$$k(an^i + bn^{i-1} + \cdots + gn + h) = hn^i + gn^{i-1} + \cdots + bn + a,$$

which will be written as

$$k(a, b, \cdots g, h)_n = (h, g, \cdots b, a)_n.$$

$k, a, \cdots, h$, are all less than $n$, while $k>1$, $a>0$ and $h>0$. If $r, s, \cdots, v, w$, are the digits carried from one place to the next when the multiplication is done, we have the equations

$$kh = wn + a,$$
$$kg + w = vn + b,$$
$$\cdots\cdots\cdots$$
$$kb + s = rn + g,$$
$$ka + r = h.$$

In each case the digit carried is less than or equal to both the multiplier and the digit multiplied, so that $w\leqq k$ and $w\leqq h$, and so on.

The inequalities given above are assumed to hold in all that follows.

**2. Numbers with 2 digits.** All solutions for this case are given by the following result.

THEOREM 1. *For any $k$ and $a$, $k(a, b)_n=(b, a)_n$ if and only if $b=ka+r$, where $r\mid(k^2-1)a$, $0<r<k$, and $n=(1/r)(k^2-1)a+k$.*

*Proof.* To prove the sufficiency of the conditions note that $k(a, b)_n=k(a, ka+r)_n=kan+k^2a+kr=kan+(k^2-1)a+kr+a=kan+rn+a=(b, a)_n$.

To prove the necessity of the conditions note that $kb=rn+a$ and $ka+r=b$, so that $k(ka+r)=rn+a$, and hence $rn=(k^2-1)a+kr$. It follows that $r$ is a factor of $(k^2-1)a$, and $n=(1/r)(k^2-1)a+k$.

From this result it can be seen that for any $a$ and any $k>2$ there are at least two solutions: with $r=1$ and with $r=k-1$. When $k=2$ these two cases coincide. If $r$ and $a$ have a common factor then this factor also divides $b$, and the solution is simply a multiple of the solution given by the values of $a$ and $r$ without the common factor.

The following result, the proof of which is straightforward, gives most 2-digit solutions for small $n$, the smallest solution that it does not give being $5(1, 8)_{13} = (8, 1)_{13}$.

LEMMA. *If $k+1 \mid n+1$ there is a 2-digit solution in the scale of $n$ such that $k(a, b)_n = (b, a)_n$, where $a = (n-k)/(k+1)$ and $b = n-a-1$.*

We are now in a position to show for which values of $n$ there are 2-digit solutions.

THEOREM 2. *There is a 2-digit solution in the scale of $n > 3$ if and only if $n+1$ is nonprime.*

*Proof.* From Theorem 1 we have $n+1 = (1/r)(k^2-1)a+k+1$. Since $r < k+1$, $(1/r)(k^2-1)$ and $k+1$ must have a common factor, so that $n+1$ must be composite.

From the lemma we have a solution whenever $n+1$ is composite, since every composite number greater than 4 has a factor which can be taken as $k+1$ such that $2 \leqq k < n$. There are no solutions for $n=2$ and $n=3$.

**3. Extension to several digits.** Any solution can be used to give other solutions with more digits by making a symmetrical pattern based on repetition and the introduction of zeros. Different solutions for the same values of $k$ and $n$ can be combined in similar ways to give new solutions. An example will make the method clear. With $n = 11$, using $3 \times 14 = 41$ and $3 \times 28 = 82$ gives $3 \times 140280028014 = 410820082041$. It is also possible to combine solutions in the following way, still with $n = 11$. Thus $3 \times 154 = 3(140+14) = 410+41 = 451$. The only condition is that there must be no carry from one digit position to the next. In the case of 2-digit solutions this can be formalized in the following way.

THEOREM 3. *If $k(a, b)_n = (b, a)_n$ then*

$$k(a, a+b, \cdots, a+b, b)_n = (b, a+b, \cdots, a+b, a)_n,$$

*where any number of digits equal to $a+b$ may be introduced between $a$ and $b$.*

*Proof.* It is sufficient to multiply both sides by $(n^m+n^{m-1}+\cdots+n+1)$ and to note that, by Theorem 1, $a+b = (k+1)a+r < (1/r)(k-1)(k+1)a+k$ (since $r<k$) $=n$.

**4. Numbers with 3 digits.** In this case the equations are

$$k(a, b, ka+r)_n = (ka+r, b, a)_n,$$
$$(k^2-1)a+kr = sn, \quad \text{and}$$
$$(k-1)b+s = rn.$$

To exclude the solutions derived from 2-digit solutions as by Theorem 3, we add the constraint $r \neq s$.

Since $rsn = (k^2-1)ar+kr^2 = (k-1)bs+s^2$, $k-1 \mid kr^2-s^2$, and hence $k-1 \mid r^2 - s^2$. Since $n = ((k^2-1)a+kr)/s$, just those values of $a$ which make $n$ an integer are required. Expressed as a congruence, the condition is $(k^2-1)a+kr \equiv 0 \pmod{s}$.

If $a$ satisfies this congruence then so does $a+s/(s, k^2-1)$; that is, each solution is one of a set of solutions that form an arithmetic progression with common difference $s/(s, k^2-1)$. So, putting $a=sx/(s, k^2-1)+y$, where $x\geqq 0$, the solution depends on $y$, which can be restricted to the range $0\leqq y<s/(s, k^2-1)$.

Here it is necessary to recall the following result. For any $A$, $B$, and $M$, the congruence $AX+B\equiv 0 \pmod{M}$ has just one solution in the range $0\leqq X<M/B$ if $(A, M)|B$, and otherwise it has no solution. (See, for example, Hardy and Wright, Theory of Numbers, Chapter 5.)

From this it follows that $y$ is defined uniquely as the solution of $(k^2-1)y+kr \equiv 0 \pmod{s}$, provided that $(s, k^2-1)|kr$, which is equivalent to $(s, k^2-1)|r$ since $k^2-1$ and $k$ can have no common factor, and otherwise there is no solution. Since $a>0$, the condition $xy\neq 0$ must be added.

Finally, the value of $b$ is to be determined, and we have $b=(rn-s)/(k-1)$. This gives the last condition, that $k-1|rn-s$. By expanding $rn-s$ it can be shown that this condition is equivalent to $s|r(k+1)y+r^2+(r^2-s^2)/(k-1)$, but this is no easier to apply than the original, and the problem has not proved amenable to further analysis. What we have is the following result.

THEOREM 4. *All solutions for 3-digit numbers, excluding those obtained from 2-digit solutions, are given as follows: For any $r$, $s>0$, $r\neq s$, any $k$ such that $k-1|r^2-s^2$, $k>r$ and $s$, and $(s, k^2-1)|r$, let $y$ be the solution of $(k^2-1)y+kr \equiv 0 \pmod{s}$, with $0\leqq y<s/(s, k^2-1)$; then for any $x$ such that $xy\neq 0$, let*

$$a = sx/(s, k^2 - 1) + y \qquad \textit{and} \qquad n = ((k^2 - 1)a + kr)/s.$$

*Then, if and only if $k-1|rn-s$, we have the solution*

$$k(a, (rn - s)/(k - 1), ka + r)_n.$$

*Proof.* That the conditions are necessary can be seen from the preceding discussion. That they are sufficient can be confirmed by substituting for $a$ and $n$, and carrying out the multiplication.

**5. What is a parametric solution?** The question arises whether this can be considered a parametric solution of the problem. In its narrowest sense the term could be taken to mean a solution in which the parameters are free to take any values independently, and to each set of values of the parameters corresponds a unique solution. However, this seems to be too narrow a definition, and it is acceptable to subject the parameters to simple conditions involving inequalities. In Theorem 4 examples of this are $r\neq s$ and $r<k$. Some of the other conditions of Theorem 4 are now considered in turn.

(1) $k-1|r^2-s^2$. This is the most problematical case since it raises the question whether $k$ is a third parameter with a very restricted range, or a function of $r$ and $s$ having several values. If the first interpretation is taken it seems open to argument whether the solution is considered to be parametric, and in general it could depend on the exact nature of the condition whether it was acceptable or not. If the second interpretation is taken it seems reasonable to reject the solution and allow only single-valued functions of parameters.

In the particular case here it is possible to introduce more basic parameters of which $r$, $s$, and $k$ are functions. For any positive integers $A$, $B$, $C$, and $D$ such that $AB$ and $CD$ have the same parity, $AB>CD$ and $BD\geqq\frac{1}{2}(AB+CD)$, we have $k=BD+1$, $r=\frac{1}{2}(AB+CD)$ and $s=\frac{1}{2}(AB-CD)$, (or with the values of $r$ and $s$ interchanged). The condition $k-1\mid r^2-s^2$ is then automatically satisfied.

(2) $(s, k^2-1)\mid r$. This raises the difficulty not present in (1) that there is not always a value of $k$ satisfying the condition for given $r$ and $s$ when the preceding conditions are also satisfied. In this situation the solution can not be considered parametric, though when the condition is a simple one the difference between this case and (1) is not great.

(3) $(k^2-1)y+kr\equiv 0 \pmod{s}$. It may be thought that to define $y$ as the solution of a congruence is also going beyond what is allowable in a parametric solution. However, given the condition (2), $y$ is defined uniquely and so can be considered as a function of $r$, $s$, and $k$. This definition of $y$ is then as acceptable as those based on the more usual algebraic functions.

(4) $k-1\mid rn-s$. This is similar to (2) but with the added difficulty that a good deal of calculation of $y$, $a$, and $n$, is necessary before it can be determined whether the condition is met. It is this case above all which prevents the solution from being considered as parametric.

Theorem 4 is best considered as a method of preparing an exhaustive list of simple parametric solutions, as shown in the Table. The solutions are listed in increasing order of $r+s$, and within that for increasing $r$, and then for increasing $k$. All values of $k$ satisfying $k-1\mid r^2-s^2$ are listed, and where there is no solution the reason is given. The list has been taken far enough to show both types of failure. The smallest solution numerically is that for $r=1$, $s=5$, $k=7$ and $x=0$: $7(1, 1, 8)_{11}=(8, 1, 1)_{11}$.

It is easily seen that from this list rather more general parametric solutions can be given, such as

$$r^2(x, (r^2+1)(rx+1), r(rx+1))_{r^3(rx+1)-x},$$

for $r>1$, $x>0$; and

$$k(x, (k^2-k-2)x+(k^2-3k+1), kx+k-2)_{(k^2-1)x+k^2-2k},$$

for $k>2$, $x>0$.

A computer program which lists the simple parametric solutions has been written in FORTRAN and run on a KDF9 of English Electric-Leo-Marconi Computers at their bureau in Kidsgrove, Staffordshire. The solutions given in the Table were confirmed, taking less than five seconds to produce. The program has been used to list all solutions for $r+s<24$.

**6. An unsolved problem.** A problem that remains to be solved is whether there is any value of $n$ for which there is a 3-digit solution but no 2-digit solution. By Theorem 2 this is equivalent to finding whether there is any 3-digit solution in which $n+1$ is prime.

What appears to be true is that for $F=(r+s, k-1)/(s, r+s, k-1)$, $1<F<n+1$ and $F\mid n+1$, so that $n+1$ is composite. This is known to be true for all cases in which $r+s<24$, but no general proof has been found. It is interesting to note that, with one exception, for $r+s<24$ and $G=(r+s, k-1)/(s, k^2-1)$, $G$

TABLE

| $r$ | $s$ | $r^2-s^2$ | $k-1$ | $k^2-1$ | $(s, k^2-1)$ | Congruence | $y$ | $a$ | $n$ | Solution |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | $-3$ | 3 | 15 | 1 | $15y+4\equiv 0 \pmod 2$ | 0 | $2x$ | $15x+2$ | $4(2x, 5x, 8x+1)_{15x+2}$ |
| 2 | 1 | 3 | 3 | 15 | 1 | $15y+8\equiv 0 \pmod 1$ | 0 | $x$ | $15x+8$ | $4(x, 10x+5, 4x+2)_{15x+8}$ |
| 1 | 3 | $-8$ | 4 | 24 | 3 | — | – | — | — | *No solution:* $(s, k^2-1)\nmid r$ |
| | | | 8 | 80 | 1 | $80y+9\equiv 0 \pmod 3$ | 0 | $3x$ | $80x+3$ | $9(3x, 10x, 27x+1)_{80x+3}$ |
| 3 | 1 | 8 | 4 | 24 | 1 | $24y+15\equiv 0 \pmod 1$ | 0 | $x$ | $24x+15$ | $5(x, 18x+11, 5x+3)_{24x+15}$ |
| | | | 8 | 80 | 1 | $80y+27\equiv 0 \pmod 1$ | 0 | $x$ | $80x+27$ | $9(x, 30x+10, 9x+3)_{80x+27}$ |
| 1 | 4 | $-15$ | 5 | 35 | 1 | $35y+6\equiv 0 \pmod 4$ | 2 | $4x+2$ | $35x+19$ | $6(4x+2, 7x+3, 24x+13)_{35x+19}$ |
| | | | 15 | 255 | 1 | $255y+16\equiv 0 \pmod 4$ | 0 | $4x$ | $255x+4$ | $16(4x, 17x, 64x+1)_{255x+4}$ |
| 2 | 3 | $-5$ | 5 | 35 | 1 | $35y+12\equiv 0 \pmod 3$ | 0 | $3x$ | $35x+4$ | $6(3x, 14x+1, 18x+2)_{35x+4}$ |
| 3 | 2 | 5 | 5 | 35 | 1 | $35y+18\equiv 0 \pmod 2$ | 0 | $2x$ | $35x+9$ | $6(2x, 21x+5, 12x+3)_{35x+9}$ |
| 4 | 1 | 15 | 5 | 35 | 1 | $35y+24\equiv 0 \pmod 1$ | 0 | $x$ | $35x+24$ | $6(x, 28x+19, 6x+4)_{35x+24}$ |
| | | | 15 | 255 | 1 | $255y+64\equiv 0 \pmod 1$ | 0 | $x$ | $255x+64$ | $16(x, 68x+17, 16x+4)_{255x+64}$ |
| 1 | 5 | $-24$ | 6 | 48 | 1 | $48y+7\equiv 0 \pmod 5$ | 1 | $5x+1$ | $48x+11$ | $7(5x+1, 8x+1, 35x+8)_{48x+11}$ |
| | | | 8 | 80 | 5 | — | – | — | — | *No solution:* $(s, k^2-1)\nmid r$ |
| | | | 12 | 168 | 1 | $168y+13\equiv 0 \pmod 5$ | 4 | $5x+4$ | $168x+137$ | $13(5x+4, 14x+11, 65x+53)_{168x+137}$ |
| | | | 24 | 624 | 1 | $624y+25\equiv 0 \pmod 5$ | 0 | $5x$ | $624x+5$ | $25(5x, 26x, 125x+1)_{624x+5}$ |
| 2 | 4 | $-12$ | 4 | 24 | 4 | — | – | — | — | *No solution:* $(s, k^2-1)\nmid r$ |
| | | | 6 | 48 | 4 | — | – | — | — | *No solution:* $(s, k^2-1)\nmid r$ |
| | | | 12 | 168 | 4 | — | – | — | — | *No solution:* $(s, k^2-1)\nmid r$ |
| 4 | 2 | 12 | 4 | 24 | 2 | $24y+20\equiv 0 \pmod 2$ | 0 | $x$ | $12x+10$ | *No solution:* $k-1\nmid rn-s$ |
| | | | 6 | 48 | 2 | $48y+28\equiv 0 \pmod 2$ | 0 | $x$ | $24x+14$ | $7(x, 16x+9, 7x+4)_{24x+14}$ |
| | | | 12 | 168 | 2 | $168y+52\equiv 0 \pmod 2$ | 0 | $x$ | $84x+26$ | *No solution:* $k-1\nmid rn-s$ |

is an integer, $1<G<n+1$ and $G|n+1$. The exception is when $r=16$, $s=4$ and $k=31$, giving the solution $31(x, 128x+66, 31x+16)_{240x+124}$. In this case $G$ is not an integer, but $F=5$, which is always a factor of $240x+125$.

**7. Numbers with 4 digits.** In extending the general results to numbers with more digits the problem is that for each additional digit there is one new equation but two new variables. Particular solutions and simple parametric solutions can always be obtained from known solutions with fewer digits by the methods of Section 3. In addition, the following result shows that there is a 4-digit solution for every value of $n$ greater than 2.

THEOREM 5. *For any* $k>1$ *and* $a>0$, $k(a, a-1, ka-1, ka)_{(k+1)a}$ *is a solution.*

*Proof.* With $n=(k+1)a$, we have $k(n-ka)=ka=n-a$. Thus $k(a, a-1, ka-1, ka)_n=kan^3+k(a-1)n^2+k(ka-1)n+k^2a=kan^3+kan^2-k(n+1)(n-ka)$ $=kan^3+kan^2-(n+1)(n-a)=kan^3+(ka-1)n^2+(a-1)n+a=(ka, ka-1, a-1, a)_n$.

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# NOTE ON THE MATRIX FUNCTIONS SIN $\pi A$ AND COS $\pi A$

JERRY C. SOUTH, JR., North Carolina State University at Raleigh

The concept of a function of a square matrix is an interesting and valuable one, not only from the viewpoint of generalization of functions of a scalar variable, but also in applications to linear systems of differential equations [1, 2]. In the latter, matrix functions that arise frequently are the exponential function and the trigonometric functions.

In particular, the trigonometric functions of a matrix are found to satisfy the usual trigonometric identities except, of course, when illegal matrix operations are involved. For example, one would exclude those identities requiring negative powers of a singular matrix function.

The purpose of this note is to point out a simple criterion for determining when the matrix functions $\sin \pi A$ and $\cos \pi A$ are singular or null by inspection of the eigenvalues of $A$. The criterion turns out to be a fairly straightforward generalization of the fact that the scalar functions $\sin \pi\lambda$ and $\cos \pi\lambda$ vanish when $\lambda$ is an integer or an odd multiple of 1/2, respectively.

THEOREM. *Given an* $n\times n$ *matrix* $A$ *over the complex field. The matrix* $\sin \pi A$ *(*$\cos \pi A$*) is a singular matrix if and only if* $A$ *has at least one eigenvalue which is an integer (odd multiple of* 1/2*). If* $A$ *has* $n$ *distinct eigenvalues, all of which are integers (odd multiples of* 1/2*), then* $\sin \pi A$ *(*$\cos \pi A$*) is the null matrix.*

*Proof.* We consider the case $f(\lambda)=\sin \pi\lambda$. The proof for $g(\lambda)=\cos \pi\lambda$ is similar.

Let the minimal annihilating polynomial [1] of the matrix $A$ be denoted by

$$\psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k} \tag{1}$$

is an integer, $1<G<n+1$ and $G|n+1$. The exception is when $r=16$, $s=4$ and $k=31$, giving the solution $31(x, 128x+66, 31x+16)_{240x+124}$. In this case $G$ is not an integer, but $F=5$, which is always a factor of $240x+125$.

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*Proof.* With $n=(k+1)a$, we have $k(n-ka)=ka=n-a$. Thus $k(a, a-1, ka-1, ka)_n=kan^3+k(a-1)n^2+k(ka-1)n+k^2a=kan^3+kan^2-k(n+1)(n-ka)$ $=kan^3+kan^2-(n+1)(n-a)=kan^3+(ka-1)n^2+(a-1)n+a=(ka, ka-1, a-1, a)_n$.

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*Proof.* We consider the case $f(\lambda)=\sin \pi\lambda$. The proof for $g(\lambda)=\cos \pi\lambda$ is similar.

Let the minimal annihilating polynomial [1] of the matrix $A$ be denoted by

$$\psi(\lambda) = (\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k} \tag{1}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the distinct eigenvalues of $A$. The degree of $\psi(\lambda)$ is $m=\sum_{i=1}^{k} m_i$, $m \leqq n$. Also let

$$r(\mu) = a_0(\mu-\mu_1)^{s_1}(\mu-\mu_2)^{s_2}\cdots(\mu-\mu_l)^{s_l} \tag{2}$$

be the Lagrange-Sylvester interpolation polynomial [1], $\sum_{j=1}^{l} s_j < m$, determined by the $m$ interpolation conditions:

$$\begin{cases} r(\lambda_i) = f(\lambda_i) = \sin \pi\lambda_i \\ r'(\lambda_i) = f'(\lambda_i) = \pi \cos \pi\lambda_i \\ \cdots\cdots\cdots\cdots\cdots \\ r^{(m_i-1)}(\lambda_i) = f^{(m_i-1)}(\lambda_i) \\ \qquad = \begin{cases} (-1)^{m_i/2-1}\pi^{m_i-1}\cos \pi\lambda_i \ (m_i \text{ even}) \\ (-1)^{(m_i-1)/2}\pi^{m_i-1}\sin \pi\lambda_i \ (m_i \text{ odd}) \end{cases} \end{cases} \tag{3}$$

where $i=1, 2, \cdots, k$.

By definition [1],

$$\sin \pi A = r(A). \tag{4}$$

Thus

$$|\sin \pi A| = |r(A)| = a_0^n |A-\mu_1 I|^{s_1} |A-\mu_2 I|^{s_2} \cdots |A-\mu_l I|^{s_l}. \tag{5}$$

Now if $A$ has an integer eigenvalue $\lambda_i$, then $\sin \pi\lambda_i = 0$, and from the first of conditions (3), $r(\lambda_i)=0$. Because of (2), this implies that $\mu_j=\lambda_i$ for some $j$, and it follows that the corresponding factor in (5) vanishes, i.e.,

$$|\sin \pi A| = 0. \tag{6}$$

Conversely if $|\sin \pi A| = 0$, then from (5) at least one $\mu_j$ is an eigenvalue of $A$. But from (2), $r(\mu_j)=0$, while from the first of conditions (3), $\sin \pi\mu_j=0$, implying that $\mu_j$ must be an integer.

Finally, if $A$ has $n$ distinct eigenvalues all of which are integers, then from (3) we have the $n$ interpolation conditions ($m_i=1$, $i=1, 2, \cdots, n$):

$$r(\lambda_1) = r(\lambda_2) = \cdots = r(\lambda_n) = 0 \tag{7}$$

which yield $r(\lambda) \equiv 0$, and from (4) we have $\sin \pi A = r(A) = 0$ (the null matrix). This completes the proof.

It should be noted that an alternate proof can be given by using the Jordan canonical form $J$ of the matrix $A (A = TJT^{-1})$ and by noting that [1]: $\sin \pi A = T(\sin \pi J)T^{-1}$.

This note was written while the author was on graduate study leave from NASA, Langley Research Center.

**References**

1. F. R. Gantmakher, The Theory of Matrices. vol. I, Chelsea, New York, 1959, Chap. V.
2. J. S. Frame, Matrix Functions and Applications. Reprinted from March-July, 1964 issues of IEEE Spectrum. Parts II and IV.

# AN INEQUALITY FOR THE PERIMETER OF THE ORTHIC TRIANGLE

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It is well known that of all the triangles inscribed in a given acute-angled triangle, the orthic triangle, that is, the triangle that has for its vertices the feet of the altitudes of the given triangle, has the minimum perimeter. If the given triangle has perimeter $2s$ and the orthic triangle has perimeter $p$, Zirakzadeh [1] has proved that $p \leqq s$ with equality if and only if the given triangle is equilateral. The following short proof of this result may be of interest.

Let $ABC$ be the given triangle; denote its sides by $a$, $b$, $c$ and its angles by $\alpha$, $\beta$, $\gamma$. Let $D$, $E$, $F$ denote the feet of the altitudes of $ABC$. Then since $AF = b \cos \alpha$, $AE = c \cos \alpha$ it follows that $EF = a \cos \alpha$. Similarly $FD = b \cos \beta$, $DE = c \cos \gamma$, so that

$$p = EF + FD + DE = a \cos \alpha + b \cos \beta + c \cos \gamma.$$

Thus the inequality $p \leqq s$ is equivalent to

$$2(a \cos \alpha + b \cos \beta + c \cos \gamma) \leqq a + b + c. \tag{1}$$

If $R$ is the circumradius of $ABC$, then $a = 2R \sin \alpha$, $b = 2R \sin \beta$, $c = 2R \sin \gamma$ and (1) becomes

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \leqq \sin \alpha + \sin \beta + \sin \gamma. \tag{2}$$

It is known that (2) holds for any triangle with equality only for an equilateral triangle (for proof see [1, p. 162]). This completes the proof of Zirakzadeh's Theorem.

Supported in part by NSF grant GP-5174.

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2. A. Zirakzadeh, An exact perimeter inequality for the pedal triangle, this MAGAZINE, 39(1966) 96–99.

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*Editorial Note.* Similar comments on the paper by Zirakzadeh were made by L. Bankhoff and S. Wolfenstein.

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**ERRATA:** (May-June, 1966 issue, page 142, "On Solutions of Certain Riccati Differential Equations"). Line 10 from bottom should read "$v^2 = f/\alpha h$" instead of "$v = f/g$".

# SHORTEST PATHS WITHIN POLYGONS

R. A. JACOBSON* and K. L. YOCOM, South Dakota State University

Many times a student blindly associates extremum problems with the discipline of calculus and neglects his "common sense." In fact, the student is often unaware that sometimes a more elegant attack can be launched from a different viewpoint [1–5]. In this paper, we wish to re-emphasize this fact. We will see that not only is another approach sometimes more acceptable but that in some situations an attempt to find a solution by calculus is almost ridiculous. In particular, we wish to consider paths of minimal length within polygons. The general problem is quite nicely illustrated by the following examples:

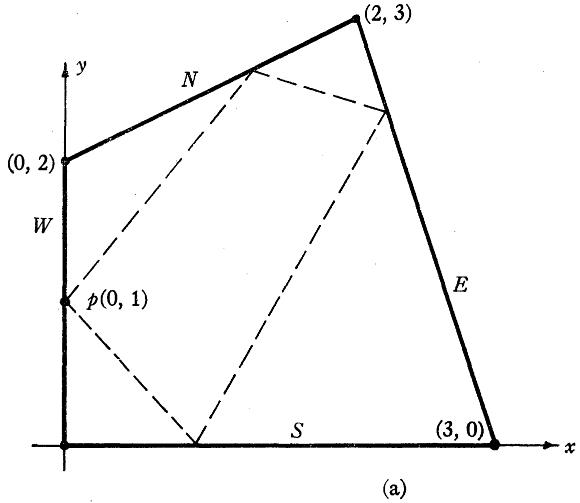



FIG. 1. (a) (b)

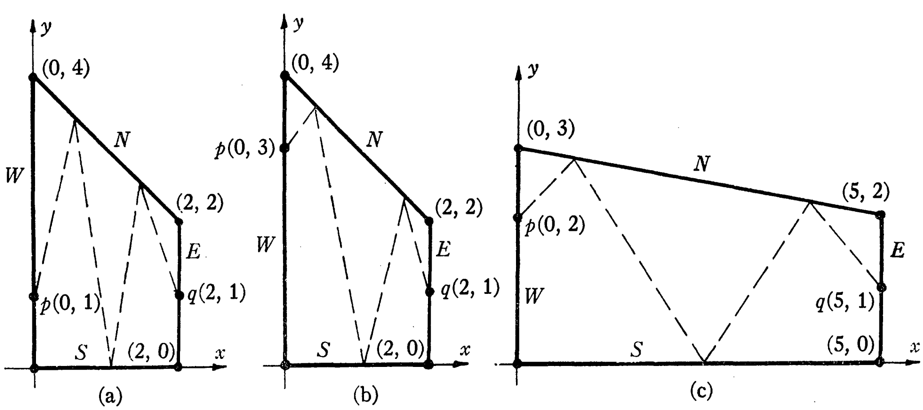



FIG. 2. (a) (b) (c)

* Presently at Houghton College.

*Problem* 1. Find the shortest closed path within each of the quadrilaterals (Figure 1) that begins at point $p$ on side $W$ and touches the sides, $S$, $E$, $N$ in succession.

*Problem* 2. Find the shortest path within each of the quadrilaterals (Figure 2) that begins at point $p$ on side $W$, touches side $N$, side $S$, side $N$ in succession and ends at point $q$ on side $E$.

*Problem* 3. Find the shortest route from $p$ to $q$ (Figure 3) which touches each of the walls $N$ and $S$ exactly once. Note that wall $N$ consists of the two line segments joining (0, 3), (2, 4), and (3, 2).

The reader is urged to investigate the difficulties in trying to solve Problem 1b using the calculus.

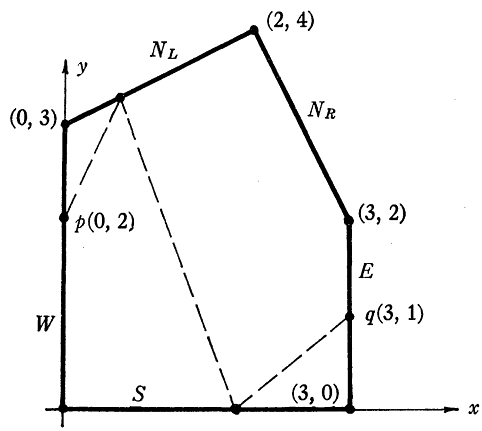



FIG. 3.

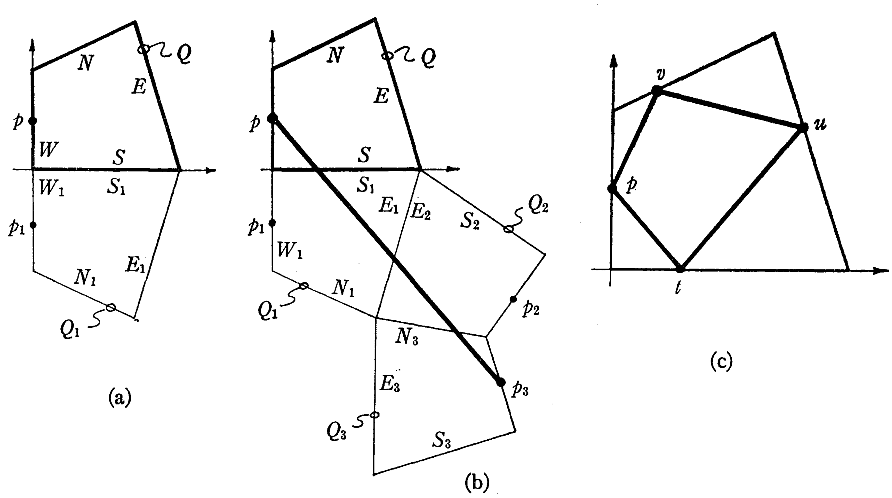


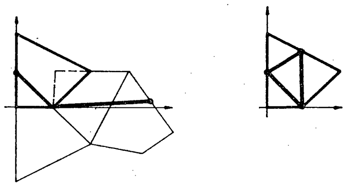

FIG. 4. (a) (b) (c)

The method of attack which we propose employs a sequence of reflections and the fact that length remains invariant. In particular, we have:

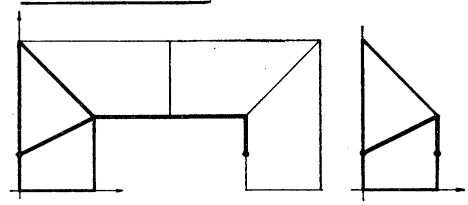
*Solution, Problem* 1a. Reflecting quadrilateral $Q$ with edges $W$, $S$, $E$, $N$ through side $S$ we obtain quadrilateral $Q_1$ with edges $W_1$, $S_1$, $E_1$, $N_1$ (Figure 4a). Successive reflections through sides $E_1$ and $N_2$ produce a polygon $P$ containing quadrilaterals $Q_2$ and $Q_3$ with edges $W_2$, $S_2$, $E_2$, $N_2$, and $W_3$, $S_3$, $E_3$, $N_3$, respectively (Figure 4b), where the image of point $p$ becomes point $p_3$ on face $W_3$. It is apparent that the straight line $pp_3$ when reflected back into quadrilateral $Q$ will yield the desired solution. In particular, the path connects points: $p$, $t(56/67, 0)$ on side $S$, $u(112/47, 87/47)$ on side $E$, and $v(14/25, 57/25)$ on side $N$ (Figure 4c).

Employing a similar technique, we find the following solutions:

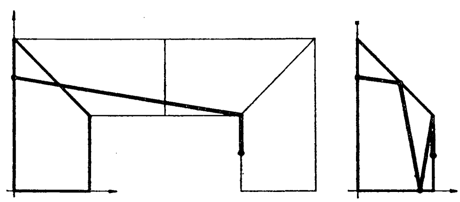
Solution, problem 1b.



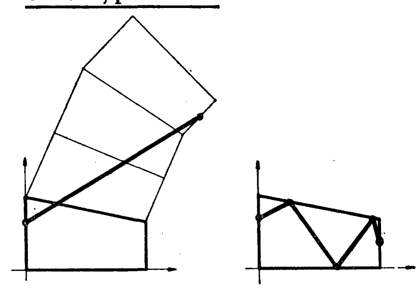
Solution, problem 2a.



Solution, problem 2b.

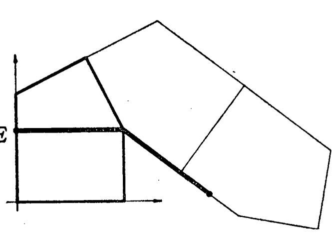


Solution, problem 2c.

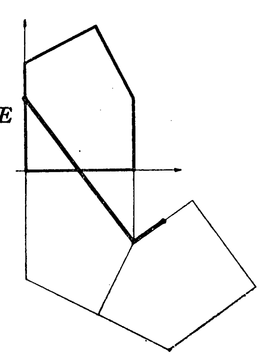


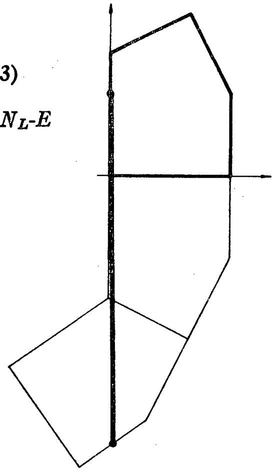
*Solution, problem* 3. In this case there are 4 possible routes. In particular, letting $N_L$ and $N_R$ denote the left and right line segments of side $N$, we have the following possibilities:

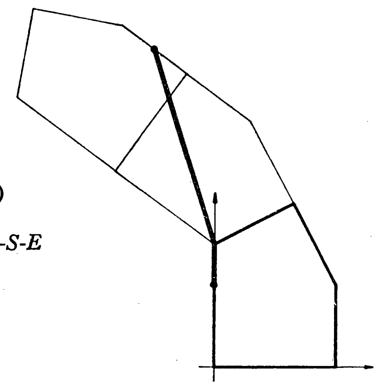
(1)
$W$-$N_R$-$S$-$E$



(2)
$W$-$S$-$N_R$-$E$






Since (1), (2), and (4) each have paths of length six, we have three solutions as shown in Figure 5.

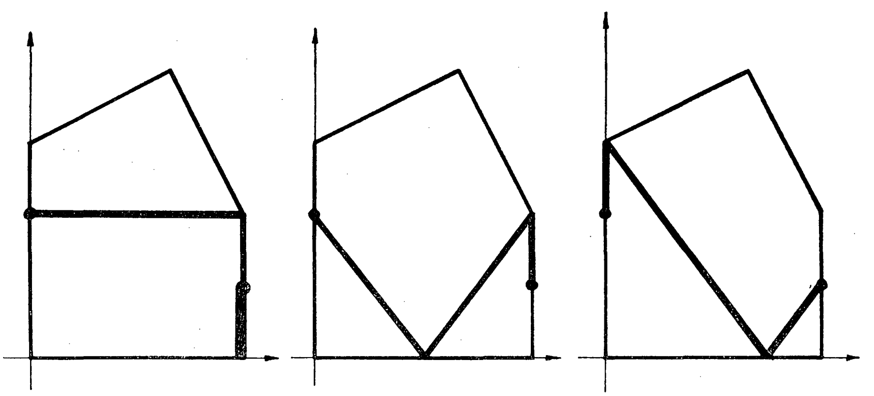


FIG. 5.

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# SOME ELEMENTARY PROPERTIES OF THE FUNDAMENTAL SOLUTION OF PARABOLIC EQUATIONS

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**1. Introduction.** In this note we shall study some elementary properties of the fundamental solution of the second order parabolic equation

$$(1) \quad Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x, t)\partial^2 u/\partial x_i \partial x_j + \sum_{i=1}^{n} b_i(x, t)\partial u/\partial x_i + c(x, t)u - \partial u/\partial t = 0,$$

where $x = (x_1, \cdots, x_n)$ is an $n$-dimensional point and $t$ a point on the real line. Before giving a precise description of the results obtained here and comparing them with those of other authors, we shall introduce some notation and make certain definitions.

Let $R_n$ denote $n$-dimensional Euclidean space and let $x$, $\xi$, etc. be elements of $R_n$ with coordinates $(x_1, \cdots, x_n)$, $(\xi_1, \cdots, \xi_n)$, etc., and let $|x| = (\sum_{i+1}^{n} x_i^2)^{1/2}$. Let $I = [t_0, T]$ be a closed interval in $R_1$ with $0 \leqq t_0 < T$, and let $t$, $\tau$, etc., be points of $I$. Let $R$ be the topological product of $R_n$ with $I$. It is customary to speak of a strip, e.g. $\tau < t < h$, and mean the topological product of $R_n$ with the interval $\tau < t < h$. Finally, we let $d\xi = d\xi_1 \cdots d\xi_n$.

A function $u(x, t)$ is said to be a solution to (1) in $R$, if it is continuous in $R$ and if in the strip $t_0 < t \leqq T$, it is twice continuously differentiable with respect to the $x$-variables and once with respect to the $t$-variables and satisfies (1) there.

Fundamental solutions arise naturally in trying to find a representation for the solution to the problem

$$(2) \qquad Lu = 0 \text{ in the strip } t_0 < t \leqq T; \qquad u(x, t_0) = \phi(x),\ x \in R_n.$$

Roughly speaking, a fundamental solution, $\Gamma$, is a function of the variables $(x, t; \xi, \tau)$ which allows one to represent the solution to the problem (2) in the form

$$u(x, t) = \int_{R_n} \Gamma(x, t; \xi, t_0)\phi(\xi)d\xi,$$

provided that $\phi$ satisfies certain additional conditions. More precisely, let us make the following definition.

DEFINITION. A function $\Gamma(x, t; \xi, \tau)$ is said to be a fundamental solution to $Lu = 0$ if:

(i) $\Gamma(x, t; \xi, \tau)$ is defined and continuous for all $(x, t)$, $(\xi, \tau) \in R$ except at $(x, t) = (\xi, \tau)$, where it is discontinuous.

(ii) $\Gamma(x, t; \xi, \tau) \equiv 0$ for $t \leqq \tau$.

(iii) There exists a positive constant, $N$, such that

$$\int_{R_n} | \Gamma(x, t; \xi, \tau) | \, d\xi \leqq N.$$

(iv) For all bounded, continuous functions $\phi$ defined on $R_n$, the function $u(x, t) \equiv \int_{R_n} \Gamma(x, t; \xi, \tau)\phi(\xi)d\xi$ is a solution to (1) in the strip $\tau \leqq t \leqq T$, and

$$\lim_{\substack{x \to x^0 \\ t \to \tau+}} u(x, t) = \phi(x^0), \qquad \text{for all } x^0 \in R_n, \quad \text{where} \quad t_0 \leqq \tau < T.$$

In the usual definition of a fundamental solution, condition (iv) is replaced by the condition

(iv′) $\Gamma(x, t; \xi, \tau)$ is a solution to (1) except at $(x, t) = (\xi, \tau)$, and for all bounded, continuous functions $\phi$ defined on $R_n$,

$$\lim_{\substack{x \to x^0 \\ t \to \tau+}} \int_{R_n} \Gamma(x, t; \xi, \tau)\phi(\xi)d\xi = \phi(x^0), \qquad \text{for all } x^0 \in R_n,$$

where $t_0 \leqq \tau < T$.

We shall see later, however, that (iv′) follows from the definition (i)–(iv). Further, every fundamental solution so far constructed satisfies (i)–(iv).

Many authors (see e.g., [1]–[4], [7], [10], [12]–[14] and the bibliography of [14] for further references) have constructed a fundamental solution $\Gamma(x, t; \xi, \tau)$ of (1) under various smoothness and growth conditions on the coefficients. Then these authors and others (see e.g., [1]–[8], [10], [12]–[14]), have investigated some of the various properties of the fundamental solution. Among other properties, the fundamental solutions so constructed have been shown to have the following elementary properties:

(i) $\Gamma(x, t; \xi, \tau) > 0$ for $t > \tau$.

(ii) $\Gamma(x, t; \xi, \tau)$ is unique.

(iii) $\int_{R_n} \Gamma(x, t; s, \theta)\Gamma(s, \theta; \xi, \tau)ds = \Gamma(x, t; \xi, \tau)$, for $t > \theta > \tau$.

The arguments used in proving (i)–(iii) have often been rather tedious (see e.g., the proof of (iii) in [3] or [14]). The object of the present note is to show that if the coefficients in $L$ satisfy certain mild growth conditions, the properties (i)–(iii) follow *a priori* from the definition of a fundamental solution. Furthermore, smoothness assumptions on the coefficients are unnecessary.

**2. Elementary properties of fundamental solutions.** We shall now make the following assumptions on the operator $L$, defined in (1), which we shall refer to collectively as assumption (A).

(A1) Let the function $a_{ij}(x, t)$, $b_i(x, t)$, $c(x, t)$, $i, j = 1, 2, \cdots, n$, be defined on $R$ and let there exist a positive constant $M$ such that

$$|a_{ij}(x, t)| \leqq M, \qquad |b_i(x, t)| \leqq M(|x| + 1), \qquad |c(x, t)| \leqq M(|x|^2 + 1),$$

$i, j = 1, 2, \cdots, n$, for all $(x, t) \in R$.

(A2) Let there exist a positive constant $\gamma$ such that

$$\gamma |\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leqq \frac{1}{\gamma} |\xi|^2,$$

for all vectors $\xi \in R_n$.

We now quote a fundamental result of Krzyżański [9] (see also [5]) in a slightly less general form than he has proven it.

THEOREM A. *Let* $L$ *satisfy assumption* (A). *If* $u(x, t)$ *is a solution to* (1) *in* $R$, *if there exists a positive constant* $m$ *such that* $|u(x, t)| \leqq m$ *for all* $(x, t) \in R$, *and if* $u(x, t_0) \geqq 0$, *then* $u(x, t) \geqq 0$ *in* $R$.

*Remark.* If in the statement of the theorem, $u(x, t_0) \equiv 0$ for all $x \in R_n$, then the theorem implies that $u(x, t) \equiv 0$ in $R$.

THEOREM 1. $\Gamma(x, t; \xi, \tau) \geqq 0$ *for* $t > \tau$.

*Proof.* Suppose there existed a point $(x', t'; \xi', \tau')$, $t' > \tau'$, such that $\Gamma(x', t'; \xi', \tau') < 0$. Then, by continuity, $\Gamma(x', t'; \xi, \tau') < 0$ for all $\xi$ satisfying $|\xi - \xi'| < \epsilon$ for some $\epsilon > 0$. Define the function

$$u(x, t) \equiv \int_{R_n} \Gamma(x, t; \xi, \tau')\phi(\xi)d\xi,$$

where $m \geqq \phi(\xi) \geqq \delta > 0$ for $|\xi - \xi'| \leqq \epsilon/2$, $\delta \geqq \phi(\xi) \geqq 0$ for $\epsilon/2 \leqq |\xi - \xi'| \leqq \epsilon$, $\phi(\xi) \equiv 0$ for $|\xi - \xi'| > \epsilon$, and $\phi$ is continuous on $R_n$. Then $Lu = 0$ and $u(x, \tau') \geqq 0$. Furthermore, $|u(x, t)| = |\int_{R_n} \Gamma(x, t; \xi, \tau')\phi(\xi)d\xi| \leqq mN$. Hence, by Theorem A, $u(x, t) \geqq 0$ in the strip $\tau' \leqq t \leqq T$. But this implies

$$0 \leqq u(x', t') = \int_{R_n} \Gamma(x', t'; \xi, \tau')\phi(\xi)d\xi = \int_{|\xi - \xi'| < \epsilon} \Gamma(x', t'; \xi, \tau')\phi(\xi)d\xi$$

$$\leqq \int_{|\xi - \xi'| \leqq \epsilon/2} \Gamma(x', t'; \xi, \tau')\phi(\xi)d\xi \leqq \delta \int_{|\xi - \xi'| \leqq \epsilon/2} \Gamma(x', t'; \xi, \tau')d\xi < 0,$$

which proves the theorem.

THEOREM 2. *The fundamental solution is unique.*

*Proof.* Suppose there existed two fundamental solutions, $\Gamma(x, t; \xi, \tau)$ and $Z(x, t; \xi, \tau)$. Form the function

$$w(x, t) \equiv \int_{R_n} [\Gamma(x, t; \xi, \tau) - Z(x, t; \xi, \tau)]\phi(\xi)d\xi,$$

where $\phi$ is an arbitrary, bounded, continuous function on $R_n$. Then $Lw = 0$, $w(x, \tau) = 0$ for all $x \in R_n$, and $|w(x, t)| \leqq \int_{R_n} [|\Gamma(x, t; \xi, \tau)| + |Z(x, t; \xi, \tau)|] |\phi(\xi)| d\xi \leqq 2mN$, where $m$ is a positive constant such that $|\phi(\xi)| \leqq m$ for all $\xi \in R_n$. Hence, $w \equiv 0$ in the strip $\tau \leqq t \leqq T$, which implies

$$\int_{R_n} [\Gamma(x, t; \xi, \tau) - Z(x, t; \xi, \tau)]\phi(\xi)d\xi = 0$$

for all bounded and continuous functions $\phi$ defined on $R_n$. Thus, $\Gamma(x, t; \xi, \tau) \equiv Z(x, t; \xi, \tau)$.

THEOREM 3. $\int_{R_n}\Gamma(x, t; s, \theta)\Gamma(s, \theta; \xi, \tau)ds=\Gamma(x, t; \xi, \tau)$ *for* $\tau<\theta<t$.

*Proof.* Consider the function

$$u(x, t) \equiv \int_{R_n} \Gamma(x, t; \xi, \tau)\phi(\xi)d\xi,$$

where $\phi$ is an arbitrary bounded, continuous function defined on $R_n$. Then $u(x, t)$ is the unique solution to the problem $Lu=0$, $\tau<t\leqq T$; $u(x, \tau)=\phi(x)$.

Now let $\theta$, $\tau<\theta<t$, be arbitrary. The function

$$w(x, t) \equiv \begin{cases} \int_{R_n} \Gamma(x, t; s, \theta)u(s, \theta)ds \\ u(x, \theta), \end{cases}$$

is the unique solution to the problem $Lw=0$, $\theta<t\leqq T$; $w(x, \theta)=u(x, \theta)$. Thus, in the strip $\theta\leqq t\leqq T$, $u(x, t)\equiv w(x, t)$. But this yields for $t>\theta$,

$$\begin{aligned} u(x, t) &= \int_{R_n} \Gamma(x, t; \xi, \tau)\phi(\xi)d\xi = w(x, t) = \int_{R_n} \Gamma(x, t; s, \theta)u(s, \theta)ds \\ &= \int_{R_n} \Gamma(x, t; s, \theta)ds \int_{R_n} \Gamma(s, \theta; \xi, \tau)\phi(\xi)d\xi \\ &= \int_{R_n} \phi(\xi)d\xi \int_{R_n} \Gamma(x, t; s, \theta)\Gamma(s, \theta; \xi, \tau)ds, \end{aligned}$$

where the interchange of the order of integration is permissible, since the first iterated integral in this set of equalities is absolutely convergent. Thus

$$\int_{R_n} [\Gamma(x, t; \xi, \tau) - \int_{R_n} \Gamma(x, t; s, \theta)\Gamma(s, \theta; \xi, \tau)]\phi(\xi)d\xi = 0,$$

for all bounded and continuous functions $\phi$ defined on $R_n$, from which it follows that $\Gamma(x, t; \xi, \tau)-\int_{R_n}\Gamma(x, t; s, \theta)\Gamma(s, \theta; \xi, \tau)ds=0$ for $\tau<\theta<t$.

COROLLARY. *As a function of* $(x, t)$ *and for* $(\xi, \tau)$ *fixed,* $\Gamma(x, t; \xi, \tau)$ *satisfies* $L\Gamma(x, t; \xi, \tau)=0$ *except at* $(x, t)=(\xi, \tau)$.

THEOREM 4. $\Gamma(x, t; \xi, \tau)>0$ *if* $t>\tau$.

*Proof.* Suppose there existed a point $(x', t'; \xi', \tau')$, $t' > \tau'$, such that $\Gamma(x', t'; \xi', \tau')=0$. Consider the function $u(x, t)\equiv\Gamma(x, t; \xi', \tau')$ in the cylinder $C$: $\{|x'-x| \leqq R, \tau'+\epsilon<t\leqq t'\}$, where $0<\epsilon<t'-\tau'$ and $R$ are arbitrary positive numbers. Then $u(x, t)\geqq 0$ in $C$ and $u(x', t')=0$. Hence, $u(x, t)\equiv 0$ in $C$ by the maximum-minimum principle. (See Nirenberg [11], who assumed the coefficients to be continuous, but the result still holds in our case.) $\epsilon$ was arbitrary so we have $u\equiv 0$ in $\tau'<t\leqq t'$, $|x'-x| \leqq R$. $R$ was also arbitrary so that we obtain $u\equiv 0$ in the strip $\tau'<t\leqq t'$. Since $u(x, \tau')=0$ by definition, we obtain $u\equiv 0$ in the

strip $\tau' \leqq t \leqq t'$. But this contradicts the fact that $\Gamma(x, t; \xi', \tau')$ is discontinuous when $(x', t') = (\xi', \tau')$.

*Concluding remark.* Note that the use of Theorem A could be replaced by any theorem to the effect that

$$Lu = 0, \qquad u(x, t_0) \geqq 0 \quad \text{and} \quad |u(x, t)| \leqq m,$$

$m$ a positive constant, implies $u(x, t) \geqq 0$, where the coefficients in $L$ satisfy growth estimates of the form

$$|a_{ij}(x, t)| \leqq M(|x|^k + 1), \qquad |b_i(x, t)| \leqq M(|x|^l + 1),$$
$$|c(x, t)| \leqq M(|x|^p + 1),$$

$i, j = 1, 2, \cdots, n$, where $M$, $k$, $l$, and $p$ are nonnegative real numbers. Of course, in this case assumption (A) must be modified appropriately. Thus, for example, we could have used Theorem 8 for paragraph 1, in Il'in, Kalashnikov and Oleinik [6] and arrived at exactly the same results.

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# ON APPROXIMATING POLYGONS BY RATIONAL POLYGONS

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A polygon is said to be rational if all its sides and diagonals are rational, and I. J. Schoenberg has posed the interesting problem, "Can any given polygon be approximated as closely as we like by a rational polygon?" Several authors have discussed aspects of this problem [**1**, **2**, **3**, **4**, **7**], some of them also requiring the area of the approximating polygon to be rational. This fact led us to consider the difference between rational polygons with and without rational area, and we found Theorem 1.

In Theorem 2 we give a new result on Schoenberg's problem for cyclic polygons with given angle ratios. All integer sided triangles with given angle ratios were determined in [**5**], and by using the same methods one could find all rational cyclic polygons with given angle ratios.

For each polygon $\mathcal{P}$ we write $\tau(\mathcal{P})$ for the set of all triangles $\mathfrak{T}$ formed by producing the sides and diagonals of $\mathcal{P}$.

THEOREM 1. *A polygon $\mathcal{P}$ is rational (with rational area) iff every triangle of $\tau(\mathcal{P})$ is rational (with rational area).*

*Proof.* It was shown by L. N. M. Carnot in 1803 (cf. [**6**] p. 217) that the segments formed by the intersection of the diagonals of a quadrilateral $\mathcal{Q}$ are expressible rationally in terms of the sides and diagonals of $\mathcal{Q}$. The same result holds for the intersections of the sides, and both facts can be proved by using the cosine rule several times. Moreover they show that $\mathcal{Q}$ is rational iff every triangle of $\tau(\mathcal{Q})$ is rational.

If $\mathcal{P}$ is any polygon and $\mathfrak{T}\in\tau(\mathcal{P})$, then any two sides of $\mathfrak{T}$ each contain two vertices of $\mathcal{P}$, and these four vertices form a quadrilateral contained in $\mathcal{P}$. Hence it follows by the first paragraph that $\mathcal{P}$ is rational iff every $\mathfrak{T}\in\tau(\mathcal{P})$ is rational.

Next suppose that $\mathcal{P}$ is rational, and let $\mathfrak{T}_i\in\tau(\mathcal{P})$. Then $\Delta_i=\frac{1}{2}a_ib_i \sin A_i$, where $\Delta_i$ is the area, $a_i$ and $b_i$ are (rational) sides, and $A_i$ is an angle of $\mathfrak{T}_i$. By the cosine rule, $\cos A_i$ is rational and so $\Delta_i^2$ is rational. It follows that if $\mathfrak{T}_i$, $\mathfrak{T}_j$ are two triangles of $\tau(\mathcal{P})$ with an angle $A_i$ in common then $\Delta_i\Delta_j$ is rational. By using this argument repeatedly we see that $\Delta_i\Delta_j$ is rational for all $\mathfrak{T}_i$, $\mathfrak{T}_j\in\tau(\mathcal{P})$. Now the area $\Delta$ of $\mathcal{P}$ is a sum of areas of triangles $\mathfrak{T}_i$ belonging to $\tau(\mathcal{P})$, say $\Delta=\sum\Delta_i$. Multiplying this equation by any $\Delta_j$ shows that $\Delta$ is rational iff $\Delta_j$ is rational, and this completes the proof of Theorem 1.

In [**1**] J. H. J. Almering shows that if $ABC$ is a plane rational triangle, the set of points $P$ such that $PABC$ is a rational quadrilateral is dense in the plane. By virtue of Theorem 1, we may now add that $ABC$ has rational area iff $PABC$ has rational area.

Let $A$ be a point on a circle $\mathcal{C}$ with centre $O$ and radius $r$, where $r^2$ is rational. Further let $\sigma(\mathcal{C})$ be the set of points $P$ on $\mathcal{C}$ with $r \tan \frac{1}{4}\phi$ rational, where $\phi$ is $\angle AOP$. It is well known that $\sigma(\mathcal{C})$ is dense in $\mathcal{C}$ and that the distance between any two points of $\sigma(\mathcal{C})$ is rational. Since the area of any polygon with vertices on $\mathcal{C}$ is rational iff $r$ is rational, this example provides a good illustration of Theorem 1.

THEOREM 2. (i) *The set of all cyclic $n$-gons with one tangential angle and $n-2$ interior angles in given rational ratios is dense in the set of all such cyclic $n$-gons.* (ii) *When $n$ is odd the same result holds for $n-1$ interior angles.*

*Proof.* Let $\mathcal{P}$ be a polygon of the type considered. We may assume that $\mathcal{P}$ has vertices $P_0, P_1, \cdots, P_{n-1}$ on $\mathcal{C}$. By hypothesis there is an angle $\phi$ and positive integers $i_\nu$ such that $i_\nu\phi$ is $\angle P_{\nu-1}P_\nu P_{\nu+1}$ for $1 \leqq \nu \leqq n-2$, whilst $i_0\phi_\nu$ is the angle between $P_0P_1$ and the tangent to $\mathcal{C}$ at $P_0$. For $0 \leqq \nu \leqq n-2$ let $2\theta_\nu$ be $\angle P_\nu O P_{\nu+1}$, and let $2\theta_{n-1}$ be $\angle P_{n-1}OP_0$. Then

$$\theta_0 = i_0\phi, \tag{1}$$

$$\theta_{\nu-1} + \theta_\nu = \pi - i_\nu\phi \qquad \text{for } 1 \leqq \nu \leqq n-2, \tag{2}$$

and

$$\theta_0 + \theta_1 + \cdots + \theta_{n-1} = \pi. \tag{3}$$

For part (i) of the theorem we use (1) and (2), but for part (ii) we use (2) and (3). In either case we find that each $\theta_\nu$ is of the form $j_\nu\phi + k_\nu\pi$, where $j_\nu$, $k_\nu$ are integers.

Now $\mathcal{P}$ is determined by $\phi$. We choose $\psi$ as close as we like to $\phi$ so that $r \tan \frac{1}{2}\psi$ is rational. This choice is possible because $\tan x$ is continuous. Then $r \sin \psi$ and $\cos \psi$ are rational, and hence $r \sin j\psi$ and $\cos j\psi$ are rational for all integers $j$. Replacing $\phi$ by $\psi$ makes $r \sin \theta_\nu$ rational for each $\nu$, and hence provides a rational cyclic $n$-gon approximation for $\mathcal{P}$.

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**Editorial Note**

Since the appearance of A. M. Vaidya's article [6], Professors N. J. Fine and E. Szekeres independently have submitted papers enlarging on Vaidya's result. It turns out that their results as well as that of Vaidya were anticipated in a paper by de Bruijn [2]. Let $C$ be a set of integers. Two subsets $A$ and $B$ of $C$ are said to be complementing subsets of $C$ in case every $c \in C$ is uniquely represented in the sum

$$C = A + B = \{x \mid x = a + b,\, a \in A,\, b \in B\}.$$

In [2] de Bruijn characterizes all pairs of complementing subsets of $N$, the set

THEOREM 2. (i) *The set of all cyclic n-gons with one tangential angle and $n-2$ interior angles in given rational ratios is dense in the set of all such cyclic n-gons.* (ii) *When n is odd the same result holds for $n-1$ interior angles.*

*Proof.* Let $\mathcal{P}$ be a polygon of the type considered. We may assume that $\mathcal{P}$ has vertices $P_0, P_1, \cdots, P_{n-1}$ on $\mathcal{C}$. By hypothesis there is an angle $\phi$ and positive integers $i_\nu$ such that $i_\nu\phi$ is $\angle P_{\nu-1}P_\nu P_{\nu+1}$ for $1 \leqq \nu \leqq n-2$, whilst $i_0\phi_\nu$ is the angle between $P_0P_1$ and the tangent to $\mathcal{C}$ at $P_0$. For $0 \leqq \nu \leqq n-2$ let $2\theta_\nu$ be $\angle P_\nu OP_{\nu+1}$, and let $2\theta_{n-1}$ be $\angle P_{n-1}OP_0$. Then

$$\theta_0 = i_0\phi, \tag{1}$$

$$\theta_{\nu-1} + \theta_\nu = \pi - i_\nu\phi \qquad \text{for } 1 \leqq \nu \leqq n-2, \tag{2}$$

and

$$\theta_0 + \theta_1 + \cdots + \theta_{n-1} = \pi. \tag{3}$$

For part (i) of the theorem we use (1) and (2), but for part (ii) we use (2) and (3). In either case we find that each $\theta_\nu$ is of the form $j_\nu\phi + k_\nu\pi$, where $j_\nu$, $k_\nu$ are integers.

Now $\mathcal{P}$ is determined by $\phi$. We choose $\psi$ as close as we like to $\phi$ so that $r \tan \frac{1}{2}\psi$ is rational. This choice is possible because $\tan x$ is continuous. Then $r \sin \psi$ and $\cos \psi$ are rational, and hence $r \sin j\psi$ and $\cos j\psi$ are rational for all integers $j$. Replacing $\phi$ by $\psi$ makes $r \sin \theta_\nu$ rational for each $\nu$, and hence provides a rational cyclic $n$-gon approximation for $\mathcal{P}$.

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$$C = A + B = \{x \mid x = a + b, a \in A, b \in B\}.$$

In [2] de Bruijn characterizes all pairs of complementing subsets of $N$, the set of nonnegative integers. In [1] and [3] he considers the structure of complementing subsets of $I$, the set of all integers, and gives sufficient conditions that a pair $A$, $B$ be complementing subsets of $I$. Long [4] reviewed results of de Bruijn and used the same approach to characterize all pairs of complementing subsets of the set $N_n = \{0, 1, \cdots, n-1\}$ for $n \geqq 1$. He also gave sufficient conditions for constructing complementing subsets of $N$ and $I$ using complementing subsets of $N_n$ and proved some interesting results concerning the number of complementing subsets of $N_n$. Moser [5] considered the problem of representing every element in the set $N_{n+1}$ in the form $a+b$ with $a \in A$ and $b \in B$ but without the requirement that $A+B=N_{n+1}$. He also gives references to articles by Erdös and Narkiewicz.

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# BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

*Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.*

*Elementary Mathematics: Its Structure and Concepts.* By Margaret F. Willerding. Wiley, New York, 1966. xi+298 pp. $6.95.

Over the past few years many books have been written to serve as textbooks for introductory courses in mathematics. The primary objective of these books has been to provide teachers and prospective teachers with the mathematics background necessary to teach the material of the elementary school curriculum both as it is today and as it may change in the future. This reviewer has recently completed a year of study designed to prepare him as an instructor of in-service classes for elementary school teachers and has been examining many texts. He personally feels that *Elementary Mathematics* by Margaret F. Willerding is the best of those many texts he reviewed. This textbook is the result of five years of experimenting with mathematics training for teachers of elementary school mathematics and in-service classes for elementary school teachers.

The exposition is clear, uncluttered, and easy to understand even for a person with little or no mathematical background. The vocabulary in her text is more than appropriate for elementary school teachers without being too wordy. Definitions are precise and correct. The author has enough rigor but keeps her generalizations and proofs relatively short and to the point. There are many of nonnegative integers. In [1] and [3] he considers the structure of complementing subsets of $I$, the set of all integers, and gives sufficient conditions that a pair $A$, $B$ be complementing subsets of $I$. Long [4] reviewed results of de Bruijn and used the same approach to characterize all pairs of complementing subsets of the set $N_n = \{0, 1, \cdots, n-1\}$ for $n \geqq 1$. He also gave sufficient conditions for constructing complementing subsets of $N$ and $I$ using complementing subsets of $N_n$ and proved some interesting results concerning the number of complementing subsets of $N_n$. Moser [5] considered the problem of representing every element in the set $N_{n+1}$ in the form $a+b$ with $a \in A$ and $b \in B$ but without the requirement that $A+B=N_{n+1}$. He also gives references to articles by Erdös and Narkiewicz.

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Topics included are sets, logic, whole numbers, systems of numeration, the algorithms, informal geometry, number sentences, topics in number theory, fractions, the integers, the rational numbers, the real numbers, and mathematical systems. I particularly liked the author's treatment of logic and informal geometry, two topics which are often left out of textbooks for in-service classes.

The book is recommended for a one-semester three unit course or as a sequence of two one-semester courses of two units each semester.

E. O. GAROT, San Jose State College

*Counterexamples in Analysis*. By Bernard R. Gelbaum and John M. H. Olmsted. Holden-Day, Inc., San Francisco, 1964. xxiv+194 pp. $7.95.

This book should be in the library of every calculus teacher. Such a compilation of examples and counterexamples illustrating the main facts of analysis has long been needed. The book is just about what the title says it is, although some of the counterexamples might be called "examples" by persons who had neglected to read the second paragraph of the preface.

The book is divided into two parts: Part I, "Functions of a Real Variable," and Part II, "Higher Dimensions." Part I consists of eight chapters devoted, respectively, to "The Real Number System," "Functions and Limits," "Differentiation," "Riemann Integration," "Sequences," "Infinite Series," "Uniform Convergence," and "Sets and Measure on the Real Axis." Part II consists of five chapters devoted, respectively, to "Functions of Two Variables," "Plane Sets," "Area," "Metric and Topological Spaces," and "Function Spaces." Each of these thirteen chapters begins with a concise, accurate exposition of basic definitions and notation, along with a few of the crucial theorems. This exposition is followed by the counterexamples themselves, ranging from only nine in some chapters, to as many as 41 in the chapter on measure.

Among the counterexamples (or examples?) to be found are a one-to-one correspondence between two intervals that is nowhere monotonic, Kolmogorov's solution of Hilbert's seventh problem, a collection of sequences such that the upper limit of their sum exceeds the sum of their upper limits, two convergent series whose Cauchy product series diverges, a non-measurable plane set having at most two points in common with any line, Besicovitch's solution of the Kakeya problem, a separable space with a non-separable subspace, and two semicontinuous functions whose sum is not semicontinuous.

The table of contents is a detailed analysis of the whole book, covering 17 pages, and listing each example in turn. There is a good bibliography of 53 entries supplying material for further study, more elaborate examples, and some omitted proofs. There is a useful glossary of special symbols and a satisfactory index.

There are a few trivial misprints on pages 5, 12, 19 and 45; the word "delineate" on page 9 is perhaps inappropriate; the organization of pages 42–3, excellent illustrative examples and most of these relate to practical situations in the present era. The included exercises are excellent with answers supplied for selected exercises. The exercises are also good in that they have some questions that will challenge every member in the class.

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R. H. BROWN, Washington College

*Percentage Baseball.* By Earnshaw Cook. M.I.T. Press, Cambridge, Massachusetts, 1966. 416 pp. $9.95.

From the fly-leaf, we gather that after retiring from his career as a metallurgist Mr. Cook settled down to his first love, baseball, and is trying to set the game straight.

It may be a long time before the men of baseball are convinced he has set them straight, in spite of many compelling arguments. Mr. Cook's strategies are based on a statistical study of prodigious amounts of data; 750,000 BPF (batter faces pitcher) in the notation of the book. This represents the complete history of major league baseball during the 1950's. The data are assembled to analyze the strategy that is used and should be used in baseball.

If one is at first shocked by the strategies, he must also be impressed by the fact that the author has correctly predicted the outcome of several major league races after the season has been one-third played. This gives great incentive to seek out the strategies his methods suggest.

The most striking strategy change concerns pitchers. The recommendations include: start the game with a relief pitcher, replace him before he bats (pinch hitter), next put in the "starting" pitcher, pull the starter on his first turn to bat unless he is "doing well" and on his second turn to bat unless the team has a comfortable lead. It seems that the usual "starter" is destined to be knocked out of the box after the fifth inning anyway (70% of all games) so why not pull him before it happens? One can scarcely visualize a Marichal or Koufax tolerating this, to say nothing of their fans' reaction, even if it worked.

The batting strategy is more appealing; put batters up exactly in order of ability, first to last. This was tried briefly by Bobby Bragan in 1956 at Pittsburgh and this attempt receives considerable discussion in the book. The methods of evaluating hitters naturally receive great attention.

The most serious question about Mr. Cook's methods concerns his apparent assumption of independence of events. He chides the baseball buff for claiming that Doakes, a .300 hitter who is 0 for 3 for the day, is now "due for a hit." Cook claims that the odds are still about 2.5 to 1, as before, or actually worse, since Doakes' average is now down a bit. This, of course, is true if each time at bat is independent of each other. But in this case old Doakes just might be trying a little harder. For Doakes, trying a little harder may be all it takes to hit, particularly if he also believes the old saw. Perhaps this independence could be checked by testing if actual streaks exceed expected streaks.

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The book is interestingly written, and the fan who can get past the notational and probabilistic difficulties he may find should be rewarded with provocative new ideas about his game.

EDGAR SIMONS, San Jose State College

*Finite Mathematical Structures.* By John E. Yarnelle. D. C. Heath and Co., Boston, Massachusetts, 1964. vi+66 pp. $1.00 (paper).

This brief pamphlet illustrates several mathematical structures and requires no real background except elementary algebra. An excellent discussion of finite mathematical fields, including modulo arithmetics and congruences, is followed by a fine introduction to the concept of a mathematical group. Much of elementary group theory including permutation groups is presented in a very understandable way. The last chapter, a presentation of Boolean Algebras, is extremely well done. Yarnelle has apparently had much success in presenting the abstract nature of mathematics to beginning students and this work demonstrates his appreciation for the problems involved. This is not a textbook in any sense but merely a few words from a man experienced in teaching the abstract nature of mathematics to beginners. In no way are the topics mentioned above treated in depth. Elementary teachers interested in mathematics, secondary teachers, and students enrolled in a first course in modern algebra will find this pamphlet worthwhile.

P. E. RUTENKROGER, Orlando Junior College

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The author is at his expository best when discussing the traditionally taught topics in geometry, plane and analytic, in trigonometry, and in measurement. All the standard terminology and the usual precautions against mistakes are stressed and illustrated. Unfortunately, the explanation of some of the newer terms and relatively recent popular concepts in mathematics is ambiguous and in some instances erroneous. For example, the description of natural numbers is based partly on the work of Bertrand Russell and partly on the definition of von Neumann with glaring inconsistencies. The definition of one-to-one correspondence is misleading and the related concept of equivalent sets is nowhere defined. Omitting zero from the set of natural numbers leads to the omission of the additive identity element in the system of natural numbers. Venn diagrams are confused with Euler circles.

Except for the few faults mentioned and some others which the careful reader will readily detect, the material is obviously organized by a master teacher and interestingly presented.

S. J. BEZUSZKA, S.J., Boston College

## AN INDUCTION FALLACY

ALBERT WILANSKY, Lehigh University

Realizing that everybody knows a dozen fallacious "proofs by induction", I still offer the following, since it seems to offer an element of novelty and more persuasiveness than any other I have seen.

A proof will be offered. It contains a flaw. The flaw will be removed. Assertion: *All positive integers are odd.* Proof: 1 is odd. Assume all positive integers $\leqq n$ are odd. Consider $n+1$. Now $n-1\leqq n$; hence $n-1$ is odd. But $n+1=n-1+2$; hence $n+1$ is also odd.

One day I told this to some friends. One of them said: "It would work if 0 were odd," referring to the fact that the proof fails to cover the case $n+1=2$. (Note that the proof is "correct" for $n+1>2$) So I changed the assertion to: *All non-negative integers are either* 0 *or odd.*

Now the above proof works since the case $n+1=2$ is also covered!

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ALBERT WILANSKY, Lehigh University

Realizing that everybody knows a dozen fallacious "proofs by induction", I still offer the following, since it seems to offer an element of novelty and more persuasiveness than any other I have seen.

A proof will be offered. It contains a flaw. The flaw will be removed. Assertion: *All positive integers are odd.* Proof: 1 is odd. Assume all positive integers $\leqq n$ are odd. Consider $n+1$. Now $n-1 \leqq n$; hence $n-1$ is odd. But $n+1=n-1+2$; hence $n+1$ is also odd.

One day I told this to some friends. One of them said: "It would work if 0 were odd," referring to the fact that the proof fails to cover the case $n+1=2$. (Note that the proof is "correct" for $n+1>2$) So I changed the assertion to: *All non-negative integers are either* 0 *or odd.*

Now the above proof works since the case $n+1=2$ is also covered!

# PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

*Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.*

## PROPOSALS

**635.** *Proposed by P. D. and R. L. Goodstein, University of Leicester, England.*

Show that there is a closed path along the edges of a regular dodecahedron which divides the dodecahedron into two congruent parts each of which contains a pair of opposite faces of the dodecahedron.

**636.** *Proposed by Vassili Daiev, Sea Cliff, New York.*

The greatest divisors of the form $2^k$ of the numbers of the sequence 2, 4, 6, 8, 10, 12, 14, $\cdots$ are 2, $2^2$, 2, $2^3$, 2, $2^2$, 2, $\cdots$. Find the $n$th term of this sequence.

**637.** *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Prove that a triangle is isosceles if and only if it has two equal symmedians.

**638.** *Proposed by Charles W. Trigg, San Diego, California.*

For what values of $n$ does the expanded form of $n!$ have exactly $2n$ digits?

**639.** *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let $ABCD$ be a convex quadrangle and $P$ be the intersection of diagonals $AC$ and $BD$. Let the distance of $P$ from the sides $AB=a$, $BC=b$, $CD=c$, $DA=d$ be $x$, $y$, $z$, and $t$ respectively. Prove that

$$x + y + z + t < \tfrac{3}{4}(a + b + c + d).$$

**640.** *Proposed by Leon Bankoff, Los Angeles, California.*

The semicircle $(O)$, described internally on the side $BC$ of a square $ABCD$, cuts the quadrant $BD$ of a circle with center at $A$ at a point $T$. Show that the circle $(P)$, tangent internally to the arc $BC$ at $T$ and tangent to the line $BC$ at $Q$ is also tangent to the circle on the diameter $AB$.

**641.** *Proposed by Yasser Dakkah, S. S. Boys' School, Qalqilya, Jordan.*

Prove that if

$$\sum_{i=1}^{n} x_i = S$$

and $0<x_i$ $(i=1, 2, \cdots, n)$, then

$$\sum_{i=1}^{n} \cosh x_i \geqq n \cosh S/n.$$

## SOLUTIONS

### Late Solutions

*G. L. N. Rao, J. C. College, Jamshedpur, India:* **610**; *C. R. J. Singleton, Petersham, Surrey, England:* **611, 613**; *Michael Goldberg, Washington, D. C.:* **613**.

### A Vexatious Cryptarithm

**614.** [March, 1966] *Proposed by Charles W. Trigg, San Diego, California.*

In the cryptarithm

$$V\ E\ X\ I\ N\ G = M\ A\ T\ H,$$

the $X$ doubles as a multiplication sign and each other letter uniquely represents a positive digit. $M\ A\ T\ H$ is a permutation of consecutive digits. Find the two numerical solutions.

**I.** *Solution by Monte Dernham, San Francisco, California.*

There are exactly six combinations of four consecutive positive digits, each of the six being permutable in 24 ways. Thus guided, and with the aid of a factor table, we proceed to reject all inadmissible permutations. After a reasonable amount of vexation, we succeed in rejecting 143 of them, and discover that

$$1\,8 \times 2\,9\,7 = 5\,3\,4\,6 \quad \text{and} \quad 2\,7 \times 1\,9\,8 = 5\,3\,4\,6\,.$$

**II.** *Comments and solution by Louis F. Bush and Lowell T. Van Tassel, San Diego City College, California.*

"Often a problem can be solved by exhaustion, by which we attempt to try all possibilities. This is frequently acceptable as a method to use on the computer."—*Problems for Computer Solution*, by F. Gruenberger and G. Jaffray. J. Wiley & Sons, 1965, p. xv.

The San Diego City College IBM 1620 was programmed by the solvers to exhaustively generate all solutions to $(VE)\cdot(ING)=(MATH)$, subject to (transitively), (digits) $V\neq E\neq I\neq N\neq G$; and furthermore $M$, $A$, $T$, and $H$ be a permutation of some 4 consecutive digits. The latter condition was implemented by testing for a computed product of $M\cdot A\cdot T\cdot H$ equal to either $4!$, or $5!/1!$, or $6!/2!$, etc. A prior restriction in the program was that all $(VE)\cdot(ING)$ products be tested by the above criteria iff these products did not exceed 9999 (or in effect 9876). (In a certain sense, it can be said that the computer made an exhaustive table search of, e.g., H. Zimmermann's *Rechentafel*, Berlin, 1891, by Wilhelm Ernst & Sohn. In this work, the *Productentafel* consists of some 201 pages, where each (two pages) consists of typically all products of, say, (190–199) by (01 to 100).) As an additional constraint, the computer program tested $V+E+I+N+G+M+A+T+H=45$, sum of the 9 digits.

*Results.* In approximately 20 minutes of execution time on the IBM 1620, the computer printed

$$18 \times 297 = 5346 \quad \text{and} \quad 27 \times 198 = 5346.$$

Interestingly enough, the computer also found that if we merely require that each *side* of the equation have intra-distinct digits, *and* that all digits sum to 45, we get

$$19 \times 346 = 6574$$

$$37 \times 148 = 5476$$

and three other such combinations, not meeting the use-*all*-digits criterion.

While we realize that the above solutions exhibit none of the highly-intuitive approaches usually found in deciphering such cryptarithms, we consider the computer-problem-solving procedure worthwhile educationally. It may, in the future, be the case that *MM* must either arbitrarily "rule out" such computer-bred solutions, or must consider such solutions to be of a different *genre* than the (quasi-) analytic category. Undoubtedly one does well, in problem proposals as well as solutions, to observe the spirit of the Dunkel *Problem Book*, where criteria (p. 5) include "Problems whose difficulty lies principally in laborious computations have gradually disappeared. . . . Problems best solved by table searching are proscribed."

*Also solved by Merrill Barnebey, Wisconsin State University, La Crosse; C. Berndtson, M.I.T. Lincoln Laboratory; Anton Glaser, Pennsylvania State University, Abington, Pennsylvania; Joseph D. E. Konhauser, University of Minnesota; Jean Satterlee Overholser, Oregon State University; C. C. Oursler, Southern Illinois University; Richard Riggs, Jersey City College; Jerome J. Schneider; Chicago, Illinois, Colin R. J. Singleton, Petersham, Surrey, England; Paul Sugarman, Swampscott High School, Massachusetts; and the proposer.*

### A Convex Surface Property

**615.** [March, 1966] *Proposed by Joseph Hammer, University of Sydney, Australia.*

Prove that in a three-dimensional convex surface whose volume is greater than the surface area numerically, infinitely many plane cross-sections can be found of which each area is greater than its perimeter.

*Solution by Stanley Rabinowitz, Far Rockaway, New York.*

Consider any line from a point on the surface to the point farthest away from it. Let this line be the $z$-axis, and one endpoint, the origin. Let $V$ be the volume of the surface, $S$ its surface area, $A(z)$ the area of any plane cross-section at height $z$, and $P(z)$ the perimeter of this cross-section. Then we have

$$\int_0^a A(z)dz = V$$

and

$$\int_0^a P(z)dz = S.$$

Suppose that only finitely many plane cross-sections have $A(z) > P(z)$. Since we can change the value of a function at a finite number of points without altering the value of its integral, we can redefine $A(z)$ at these points such that $A(z) \leqq P(z)$. Then for all $z$, $A(z) \leqq P(z)$. But upon integrating we find that $V<S$, a contradiction. Hence the theorem.

*Also solved by Colin R. J. Singleton, Petersham, Surrey, England; and the proposer.*

**A Reducible Fraction**

**616.** [March, 1966] *Proposed by Rosemary Griffith, Technical Operations Research, Burlington, Massachusetts.*

(a) The factor $6/(N^3-N)$ appears in the Spearman rank correlation coefficient, where $N$ is the sample size. Show that this factor can be reduced to the form $1/M$, $M$ an integer.

(b) In general, determine the restrictions on $n$ (an integer) and $m$ (a positive integer) so that the expression $(n^m-n)$ is divisible by $m!$.

*Solution by Merrill Barnebey, Wisconsin State University, La Crosse.*

(a) We have

$$\frac{6}{N^3 - N} = \frac{6}{N(N-1)(N+1)} \cdot$$

Both 2 and 3 divide the denominator as the factors are three consecutive integers. Therefore 6 divides both numerator and denominator, so the fraction can be written as $1/M$, $M$ an integer.

(b) Here

$$\begin{aligned} N^m - N &= N(N^{m-1} - 1) \\ &= N(N-1)(N^{m-2} + N^{m-3} + \cdots + N + 1). \end{aligned}$$

For $m=0, 1, 2, 3$, the proposition is true for all $N$. For $m=4$ the proposition holds for $N$ a perfect square greater than 4, i.e., $n=9, 16, 25, \cdots$ and certain other values such as $N=24$, 48 or other multiples of 4! It also holds for $N-1$ a multiple of 4! For $m=5$, we have $N^5-N=N(N-1)(N+1)(N^2+1)$. The proposition holds for all odd $N$ and for even $N\equiv 0 \pmod 8$. For $m>5$ the relationship becomes less obvious.

*Also solved by Jerome J. Schneider Chicago, Illinois (part (a)); and the proposer.*

**Lines of a Regular Polygon**

**617.** [March, 1966] *Proposed by Norman Schaumberger and Erwin Just, Bronx Community College, New York.*

A regular polygon of $2n$ sides has a unit radius and vertices $A_i$ $(i=1, 2, \cdots, 2n)$. If $a_k$ is the length of the line segment $A_{k+1}A_1$, prove that

$$\prod_{i=1}^{n-1} a_i^2/(4 - a_i^2) = 1.$$

**I.** *Solution by Colin R. J. Singleton, Petersham, Surrey, England.*

Consider the triangle $A_1A_{j+1}A_{n+1}$. Now $A_1A_{n+1}$ is a diameter of the circumcircle and is of length 2. By definition, $A_1A_{j+1}=a_j$. By symmetry with triangle $A_1A_{n+1-j}A_{n+1}$, $A_{j+1}A_{n+1}=A_1A_{n+1-j}=a_{n-j}$. By Pythagoras' Theorem $a_j^2+a_{n-j}^2=4$ or $a_{n-j}^2=4-a_j^2$. Therefore

$$\prod_{i=1}^{n-1}\frac{a_i^2}{(4-a_i^2)}=\prod_{i=1}^{n-1}\frac{a_i^2}{a_{n-i}^2}$$

$$=\prod_{i=1}^{n-1}a_i^2\Big/\prod_{i=1}^{n-1}a_i^2=1.$$

**II.** *Solution by William L. Mrozek, University of Michigan.*

The length of $a_i$ is given by the law of cosines.

$$a_i^2=2-\cos(\pi i/n)$$
$$4-a_i^2=2+\cos(\pi i/n)$$

For $(n-1)\geqq i\geqq 1$, we have

$$\cos(\pi i/n)=-\cos(\pi(n-i)/n).$$

Hence the terms cancel in pairs giving the desired result.

**III.** *Solution by W. Moser, McGill University.*

Representing the vertex $A_k$ by the complex number

$$e^{i(k-1)\pi/n}\qquad(k=1,2,\cdots,2n)$$

we see that

$$a_k^2=\left|e^{ik\pi/n}-1\right|^2=2(1-\cos k\pi/n)$$

and

$$4-a_k^2=2(1+\cos k\pi/n)=2(1-\cos(n-k)\pi/n),$$

so that

$$\prod_{k=1}^{n-1}(4-a_k^2)=\prod_{k=1}^{n-1}2(1-\cos(n-1)/n)$$

$$=\prod_{m=1}^{n-1}2(1-\cos m\pi/n)=\prod_{k=1}^{n-1}a_k.$$

*Also solved by Charles K. Brown, Westtown School, Westtown, Pennsylvania; John Burslem, St. Louis University; Stephen Hoffman, Trinity College, Connecticut; Aughtum Howard, Eastern Kentucky State College; Bruce W. King, Burnt Hills-Ballston Lake High School, New York; J. D. E. Konhauser, University of Minnesota; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Stephen Spindler, Ft. Wayne, Indiana; Gary B. Weiss, New York University School of Medicine; K. L. Yocom, South Dakota State University; and the proposers.*

**Idempotent Matrices**

**618.** [March, 1966] *Proposed by Albert Wilansky, Lehigh University.*

Let $A$ and $B$ be $n$ by $n$ idempotent $(A^2=A)$ matrices of real numbers such that $(A-B)^2=0$. Then either $A$ and $B$ have the same range or they have the same nullspace. Prove that this is true for $n=2$, 3 and false for $n\geqq 4$.

*Solution by G. N. Wollan, Purdue University.*

With $A^2=A$ and $B^2=B$ the condition $(A-B)^2=0$ is equivalent to $A+B=AB+BA$. For $n=3$, there are 3 cases to consider.

*Case* 1. $Ax=0$ for all $x$ and there is $x_0$ such that $Bx_0\neq 0$. Then $(A+B)x_0=Bx_0\neq 0=(AB+BA)x_0$. Hence this cannot occur.

*Case* 2. $\{x_1, x_2\}$ is a basis for the null space of $A$, $\{x_3\}$ is a basis for the range of $A$ and $Bx_1\neq 0$. Then $(A+B)x_i=Bx_i=(AB+BA)x_i=ABx_i$ for $i=1$, 2. So $Bx_1$ and $Bx_2$ are both in the range of $A$; i.e., each is a multiple of $x_3$. Also $(A+B)x_3=x_3+Bx_3=(AB+BA)x_3=ABx_3+Bx_3$. Hence $Bx_3$ is also in the range of $A$ since it can differ from $x_3$ only by some multiple of $x_3$. Since $\{x_1, x_2, x_3\}$ is a basis for the whole space it follows that the range of $B$ is the range of $A$.

*Case* 3. $\{x_i\}$ is a basis for the null space of $A$, $\{x_2, x_3\}$ is a basis for the range of $A$ and $Bx_1\neq 0$. Then $(A+B)x_1=Bx_1=(AB+BA)x_1=ABx_1$. Hence $Bx_1$ is in the range of $A$. Also $(A+B)x_i=x_i+Bx_i=(AB+BA)x_i=ABx_i+Bx_i$ for $i=2$, 3. Hence $Bx_i=x_i$ for $i=2$, 3 since if $Bx_i=x_i+\lambda x_1$ then $Bx_i=Bx_i+\lambda Bx_1$, so $\lambda=0$ since $Bx_1\neq 0$. It follows that the range of $B$ is the range of $A$.

A slightly simpler version of the above argument proves the assertion when $n=2$.

For $n\geqq 4$ let $\{x_1, x_2\}$ be a basis for the null space of $A$, let $\{x_3, x_4, \cdots, x_n\}$ be a basis for the range of $A$ and let $Bx_1=x_3$, $Bx_2=0$, $Bx_i=x_i+x_2$ for $i=3, 4, \cdots, n$. Then the corresponding real matrices $A=(a_{ij})$ and $B=(b_{ij})$, with $a_{11}=a_{22}=a_{ij}=0$ for $i\neq j$, $a_{ii}=1$ for $i=3, 4, \cdots, n$, $b_{31}=b_{23}=b_{24}=\cdots=b_{2n}=1$, $b_{33}=b_{44}=\cdots=b_{nn}=1$ and $b_{ij}=0$ otherwise, satisfy the conditions but do not have either the same null space or the same range.

**A Triangle with Integer Sides**

**619.** [March, 1966] *Proposed by Alan Sutcliffe, Congleton, Cheshire, England.*

In a triangle $ABC$, right-angled at $C$, the bisectors of angles $A$ and $B$ meet $BC$ and $AC$ at $D$ and $E$, respectively. If $CD=9$ and $CE=8$, what are the lengths of the sides of the triangle?

I. *Solution by Frank L. Kerr, Napa, California.*

Let $O$ be the incenter of triangle $ABC$ and let $F$ and $G$ be the feet of the perpendiculars from $O$ to $BC$ and $AC$, respectively. Then

$$\measuredangle EOC = \measuredangle ODC = 45^\circ + \tfrac{1}{2}\measuredangle B$$

$$\measuredangle ECO = \measuredangle DCO = 45^\circ.$$

Triangle $EOC$ is similar to triangle $ODC$. Hence $9/OC=OC/8$ or $OC=6\sqrt{2}$, and $OG=OF=6$. From the similarity of triangle $EOG$ and triangle $EBC$ we get $BC=24$ and from the similarity of triangle $DOF$ and triangle $DAC$ we get $AC=18$. The desired triangle has sides 18, 24, and 30.

In general, if $CE=a$, $CD=b$ where $0<a/2<b$ and $0<b/2<a$, we have:

$$BC = a(\sqrt{(2ab} + b)/(2a - b)$$

$$AC = b(\sqrt{(2ab} + a)/(2b - a)$$

and

$$AB = \sqrt{(BC^2 + AC^2)}.$$

**II.** *Solution by Leon Bankoff, Los Angeles, California.*

Using conventional notation, we have $CD=ab/(b+c)$ and $CE=ab/(a+c)$. Thus

$$CD/CE = (a + c)/(b + c)$$
$$= 9/8$$

or

$$c = 8a - 9b.$$

Then

$$c^2 = 64a^2 - 144ab + 81b^2 = a^2 + b^2$$

or

$$63a^2 - 144ab + 8b^2 = 0.$$

Solving as a quadratic in $a$ and discarding the extraneous root $a=20b/21$, we have $a=4b/3$. Then

$$c = 8a - 9b = 5b/3.$$

Since $CD=ab/(b+c)=9$, we have $b=18$, $a=24$, and $c=30$.

*Also solved by Merrill Barnebey, Wisconsin State University, La Crosse; C. Berndtson, M.I.T. Lincoln Laboratory; Charles K. Brown, Westtown School, Westtown, Pennsylvania; Mannis Charosh, Brooklyn, New York; Stephen D. Clamage, Altadena, California, Herta T. Freitag, Hollins College, Virginia; Hwa S. Hahn, State College, Pennsylvania; John Y. Hung, Kiski School, Saltsburg, Pennsylvania; J. A. H. Hunter, Toronto, Canada; Erwin Just, Bronx Community College; Bruce W. King, Burnt Hills-Ballston Lake High School, New York; Joseph D. B. Konhauser, University of Minnesota; Herbert R. Leifer, Pittsburgh, Pennsylvania; Howard Marston, St. Louis, Missouri; C. C. Oursler, Southern Illinois University; G. L. N. Rao, J. C. College, Jamshedpur, India; Stanley Rabinowitz, Far Rockaway, New York; Lawrence A. Ringenberg, Eastern Illinois University; Jerome J. Schneider; Ronald R. Schryer, Orange Coast College, California; Colin R. J. Singleton, Petersham, Surrey, England; Paul Sugarman, Swampscott High School, Massachusetts; Charles W. Trigg, San Diego, California; Gary B. Weiss, New York University School of Medicine; Hazel S. Wilson, St. Petersburg, Florida; Samuel Wolf, Linthicum Heights, Maryland; K. L. Yocom, South Dakota State University; and the proposer.*

**The Crazy Clock of Zurich**

**620.** [March, 1966] *Proposed by Daniel B. Lloyd, District of Columbia Teachers College.*

(a) The town clock in Zurich was started one evening at 6 o'clock in a manner to cause the natives to believe it to be bewitched. The hour and minute hands had been accidentally interchanged, causing the hour hand to rotate twelve times as fast as the minute hand. How soon thereafter would the hands tell exactly the correct time?

(b) The clock repairman, duly embarrassed, labored feverishly all night to correct his error. However, in the confusion, and with poor light, the clock pinions became interchanged. When the clock was started again at 6 o'clock the next morning, the hands appeared correct on the face, but the hour hand again started rotating twelve times as fast as the minute hand. However, when the town officials came later in the morning to inspect the work, the clock showed the correct time. What time was it then?

*Solution by Samuel Wolf, Linthicum Heights, Maryland.*

(a) Let $t$ equal the number of minutes after the 6 o'clock start. The hour hand of the "true" clock will coincide with the hour hand of the Zurich clock when $30+t/12=t$. The minute hand requires the same relation for coincidence. Solving for $t$, we get $t=32\ 8/11$ minutes so the Zurich clock shows the correct time at 6:32 8/11.

(b) Again, the hour hand of the "true" clock will coincide with the hour hand of the Zurich clock when $30+t/12\equiv 30+t \pmod{60}$; and for the minute hands, $t\equiv t/12 \pmod{60}$, the same requirement as for the hour hands. Solving leads to $11t/12$ is divisible by 60. The smallest value of $t$ is 65 5/11 minutes. Therefore the town's officials' inspection was made at 7:05 5/11 or $n$ 65 5/11 minutes thereafter ($n=1, 2, 3, \cdots$).

*Also solved by Merrill Barnebey, Wisconsin State University, La Crosse; John Burslem, St. Louis University; Frank L. Kerr, Napa, California; Joseph D. E. Konhauser, University of Minnesota; Jerome J. Schneider; Colin R. J. Singleton, Petersham, Surrey, England; and the proposer.*

The proposer dedicated the problem to his Uncle Sam Lloyd.

Konhauser pointed out that the problem appeared in the Cyclopedia of Puzzles, by Sam Lloyd, 1914.

**Confused Statements**

**370.** [March, 1959] *Proposed by D. L. Silverman, Greenbelt, Maryland.*

Let $xy$ denote $x$'s statement to $y$. Determine the truth or falsity of the following set of statements:

$AB$: Someone is not lied to.
$AC$: Someone lies twice.
$BA$: Someone neither lies twice nor is lied to twice.
$BC$: Someone is lied to twice.
$CA$: Someone lies and is lied to.
$CB$: Someone does not lie.

*Solution by Stanley Rabinowitz, Far Rockaway, New York.*

Consider the truth-values of the statements $AB$ and $BC$. There are only four possibilities (if these truth values exist):

(i) $AB \wedge BC$
$\Rightarrow (\sim BA) \wedge (\sim CA)$ [since someone is lied to twice]
$\Rightarrow BA$ [since $B$ neither lies twice nor is lied to twice]
contradiction

(ii) $AB \wedge (\sim BC)$
$\Rightarrow AC$ [otherwise $C$ would be lied to twice]
$\Rightarrow CB$ [since $A$ does not lie]
$\Rightarrow (\sim BA)$ [since someone lies twice]
$\Rightarrow BA$ [$C$ neither lies twice nor is lied to twice]
contradiction

(iii) $(\sim AB) \wedge BC$
$\Rightarrow (\sim AC)$ [otherwise $C$ would not be lied to]
$\Rightarrow AC$ [$A$ lies twice]
contradiction

(iv) $(\sim AB) \wedge (\sim BC)$
$\Rightarrow AC$ [otherwise $C$ would be lied to twice]
$\Rightarrow CB$ [otherwise $B$ would be lied to twice]
$\Rightarrow (\sim BA)$ [someone must lie twice]
$\Rightarrow BA$ [$C$ neither lies twice nor is lied to twice]
contradiction

Since all these cases are inconsistent, the given set of statements must be self-contradictory.

## COMMENT ON PROBLEM 586

**586.** [May, 1965, and January, 1966] *Proposed by Maxey Brooke, Sweeny, Texas.*

"Jim Clark told me that he saw a flying saucer." I told Ford.

"You can't believe a word Clark says." Ford answered.

"That's peculiar," I replied with my usual degree of truthfulness, "Clark said just the opposite about you."

What is the probability that Clark saw a flying saucer?

*Comment by Colin R. J. Singleton, Petersham, Surrey, England.*

Let us allot "truth factors" ($1=$true, $0=$false) to each fact in dispute. Next label the statements:

$S\rightarrow$Clark said he saw a flying saucer.
$C\rightarrow$Clark tells the truth.
$F\rightarrow$Ford tells the truth.
$I\rightarrow$I tell the truth.

Each of $S$, $C$, $F$, and $I$ equals 1 or 0.

We now form three equations from the three statements made:

(1) $S=I$ (Clark said it only if I tell the truth)

(2) $F \sim C = 1$ (Clark and Ford have opposite veracity)
(3) $I \sim (F \sim C) = 1$ (If I lie, Ford and Clark have opposite veracity. If I tell the truth, Fred and Clark have the same veracity.)

Whence from (3), since $F \sim C = 1$, $I = 0$. Therefore $S = 0$, Clark did not say he saw the flying saucer. This, however, does not solve the original problem. Clark may have seen it and not said so. If he is a liar, which has probability of .5, he couldn't say so if he saw it, though he may say not, which still fits the conversation given in the statement of the problem.

### COMMENT ON Q381

**Q381.** [March, 1966] *Comment by Charles W. Trigg.*

**Q381** appears as solution 121, page 82, of Bryant-Graham-Wiley, "Non-routine Problems," McGraw-Hill Book Co. (1965), except that $(\sqrt{3})^{\sqrt{2}}$ has been used instead of $(\sqrt{2})^{\sqrt{2}}$.

## QUICKIES

*From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.*

**Q393.** Sum

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{2^n} + 1}$$

[*Submitted by Murray S. Klamkin*]

**Q394.** Given a line segment $AB$ of length $2k$. Find the area of the plane ring whose outer circle goes through $A$ and $B$, and inner circle is tangent to $AB$.

[*Submitted by Vladimir F. Ivanoff*]

**Q395.** Prove that there are infinitely many primes of the form $x^2 - y^2 - 1$, $x$ and $y$ being positive integers.

[*Submitted by R. S. Luthar*]

**Q396.** Evaluate

$$I_n = \int_0^{\pi/2} \frac{\sin^2(n\theta)}{\sin\theta}\, d\theta,$$

$n$ an integer.

[*Submitted by Henry E. Fettis*]

**Q397.** Determine

$$\lim_{n\to\infty} \frac{1}{n} \sqrt[n]{((n+1)(n+2)\cdots(n+n))}.$$

[*Submitted by Murray S. Klamkin*]

**(Answers on page 281)**

## ANSWERS

**A393.** Here

$$\frac{2^n}{3^{2^n}+1} = \frac{2^n}{3^{2^n}-1} - \frac{2^{n+1}}{3^{2^{n+1}}-1}.$$

So the sum $S=1/2$.

**A394.** If $R$ and $r$ are the radii of the ring then

$$R^2 = r^2 + k^2$$

and the area of the ring $=\pi(R^2-r^2)=\pi k^2$.

**A395.** Let $x=z+1$ and $y=z-1$. Then

$$\begin{aligned} x^2 - y^2 - 1 &= (z+1)^2 - (z-1)^2 - 1 \\ &= 4z - 1. \end{aligned}$$

We know that there are infinitely many primes of the form $4z-1$ for different values of $z$.

**A396.**

$$I_n = \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos(2n\theta)}{\sin\theta}\,d\theta.$$

Hence

$$\begin{aligned} I_{n+1} - I_n &= \frac{1}{2}\int_0^{\pi/2} \frac{\cos(2n\theta) - \cos(2n+2)\theta}{\sin\theta}\,d\theta \\ &= \int_0^{\pi/2} \sin(2n+1)\theta\,d\theta \\ &= \frac{1}{2n+1}. \end{aligned}$$

Since $I_0=0$ we obtain by induction

$$I_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}, \qquad n > 0.$$

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$$\log L = \lim_{n\to\infty} \frac{1}{n}\left[\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) + \cdots + \log\left(1+\frac{n}{n}\right)\right]$$

$$\log L = \int_0^1 \log(1+x)\,dx = 2\log 2 - 1 \text{ from the definition}$$

of a definite integral. Therefore $L=4/e$.

**(Quickies on page 315)**